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ABSTRACT

This publication provides reports from several mathematics educators on the gains that have been made, and the problems still unsolved, related to curriculum activities in mathematics. Answers to certain key questions which are basic in curriculum development are explored--(1) What are the goals of teaching mathematics, (2) What mathematical ideas, skills, attitudes, and habits can be most effectively developed at a given grade level, (3) How should programs be varied to provide for different levels of ability, (4) How do we teach for transfer so that mathematical ideas will be used in solving problems, (5) What degree of rigor or mathematical precision in language and logic is appropriate at various grade levels, (6) What emphasis should be given to computational skill, (7) What is the role of the computer in the mathematics program, (8) How do we prepare teachers for the new programs, (9) How do we evaluate the effectiveness of a new mathematics program, (10) What criteria should be used in selecting instructional materials, (11) How shall the achievement of students of different ability be graded, and (12) How are students selected for different curriculum tracks. Discussions are intended to point to resolutions of these questions as well as to help determine the kind of mathematics programs available and appropriate for your school. (RP)

REVOLUTION

NCTM

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Continuing REVOLUTION in Mathematics

Reprinted from the
Bulletin of the National Association
of Secondary-School Principals
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Contents

WARREN C. SEYF'.RT, *Editor*

Second Thoughts: THE EDITOR COMMENTS

About This Issue

Although unnumbered nonmathematicians who inhabit principals' quarters in American secondary schools still use it as a phrase to enrich their curriculum incantations, "the new math," in the waning 60's, no longer frightens the timid or uninformed. The fact of the matter is, however, that the revolution in mathematics instruction is by no means over. There are many classrooms that remain unmarked by the nearby intellectual artillery, and new weapons continue to come out of our academic arsenals.

It is true that some of what was revolutionary in the 1950's has become the new tradition. ("It was much too young to die!") And it must also be said that some of the changes that have appeared in texts and in schools are more cosmetic than internal. In spite of these conditions the revolutionary spirit in mathematics persists and, significantly, has not lost its adolescent vigor as it has matured. And the successes achieved thus far have not substantially diminished the need for continued assault on the irrelevancies and illogic which, some would say, justify calling mathematics the "Queen of the Sciences."

The editor is much in debt to the scholars who in the pages that follow provide front-line reports on the gains that have been made—and the problems still unsolved—as they and their associates work to bring their instructional specialty, mathematics, up to date and beyond. He is also most grateful to the officers of the National Council of Teachers of Mathematics who responded so promptly and favorably to his proposal for collaboration. From this reaction came the appointment of a Joint Advisory Committee, whose members are listed in the Foreword. The outline of topics for this issue and the authors of the articles herein came directly from recommendations made by this committee. Thank you, ladies and gentlemen.

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Because numerous references are made to the Cambridge Conference on School Mathematics in the papers that follow, the editor decided to insert here the description of CCSM that Professor Nichols wrote as part of his article, "The Many Forms of Revolution," which starts on page 16.

Cambridge Conference on School Mathematics (CCSM)

CCSM [writes Nichols] grew from a 1963 conference of 25 mathematicians at Cambridge, Massachusetts, who met for the purpose of exploring curriculum reforms and setting goals for future mathematics in the elementary and secondary schools. Their conclusions were published in *Goals for School Mathematics.*

Goals for School Mathematics—the Report of the Cambridge Conference on School Mathematics. Boston: Houghton-Mifflin, 1963.

In essence, that conference suggested, first, an acceleration of topics so that bright students would have completed what is now the third year of college mathematics by the time they finished high school, and, second, "that the traditional drill-for-drill's-sake should be abandoned and that manipulative skills should be taught instead by the adroit introduction of problems whose primary purpose would be to illustrate mathematical concepts." Drill, CCSM says, does not teach a student when or why to use the mathematical techniques over which he has labored so long.

CCSM presented an overview of topics for the various grade levels and also suggested some means of presenting the topics. "It was felt, for example, that it was desirable to adopt the 'spiral' approach, in which every new topic is introduced early under low pressure and is then reconsidered repeatedly, each time with more sophistication and each time showing more of its interconnections with the rest of the subject."

Meeting again in 1965, CCSM decided to write some experimental units whereby their suggestions could be tested. Over 30 of these units have since been written and are in use around the country. The Conference meeting in 1966 focused on the problems of teacher training in mathematics, from which came *Goals for Mathematical Education of Elementary School Teachers.*

Goals for Mathematical Education of Elementary School Teachers—the Report of the Cambridge Conference on Teacher Training. Boston: Houghton-Mifflin, 1967.

The project is affiliated with Education Development Center, Inc., and is directed by Hugh P. Bradley.

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Foreword

It is time once more to give attention to *The Continuing Revolution in Mathematics.* Nine years ago the National Council of Teachers of Mathematics and the National Association of Secondary School Principals collaborated in the preparation of *New Developments in Secondary School Mathematics,* which was published as the NASSP BULLETIN for May 1959. In the years since then, much has happened to change instruction in mathematics; these developments merit thoughtful analysis.

The National Council of Teachers of Mathematics has again joined with us in developing a BULLETIN that concentrates on curriculum activities in mathematics. We are most grateful to members of the Joint Editorial Committee appointed by NCTM and NASSP, who helped the BULLETIN editor plan this issue, and to Kenneth E. Brown, Senior Specialist in Mathematics of the U.S. Office of Education, for his advice. The members of the Joint Editorial Committee are:

James D. Bristol, Chairman of the Mathematics Department, Shaker Heights High School, Shaker Heights, Ohio

James D. Gates, Executive Secretary, National Council of Teachers of Mathematics, Washington, D.C.

Agnes Y. Rickey, Supervisor of Mathematics, Division of Instruction, Dade County Public Schools, Miami, Florida

G. Howard Schofstal, Principal, Annapolis Junior High School, Annapolis, Maryland

Harold L. Secord, Principal, T. C. Williams High School, Alexandria, Virginia

We hope this joint publication will have wide distribution, for it gives promise of usefulness to principals, supervisors, university instructors, classroom teachers, and interested parents.

Ellsworth Tompkins, *Executive Secretary*
National Association of
Secondary School Principals

There have been some exciting changes in mathematics instruction in the last decade and there are more to come. The president of the National Council of Teachers of Mathematics reminds us, however, that some older questions about learners and learning have remained invariant under this transformation.

The New Mathematics in Our Schools

Donovan A. Johnson

Today, the spirit of innovation is characteristic of mathematics education; new programs continue to alter traditional content and classroom activities of mathematics at all school levels. The questions of where these innovations are leading and how they should be instituted are now a matter of great concern to educators who have responsibility for developing local school programs. Thus, it is an appropriate time for the National Council of Teachers of Mathematics and the National Association of Secondary School Principals to join forces in discussing school mathematics programs.

The mathematical competence of many of our present students is certainly not as high as it could be under optimum conditions. Problems caused by large classes, great differences in aptitude, inadequate materials, unprepared teachers, and unsatisfactory course content need to be solved. The mathematics programs of our schools urgently need to be evaluated and strengthened. The wealth of resources in the form of new texts, new topics, new materials, new emphasis, and new methods need to be put to use in every school.

Donovan A. Johnson, president of the National Council of Teachers of Mathematics 1966-1968, is professor of mathematics education at the University of Minnesota.

Just as the nature of our society is continuously changing, so the mathematics curriculum must be constantly re-examined and revised in the light of new societal and intellectual conditions.

As you read the articles which follow, try to find answers to these key questions which are basic in curriculum development:

1. *What are the goals of teaching mathematics?* What is the role of mathematics in the life of pupils of different abilities and cultures? Are we teaching mathematics for vocational needs, for improved citizenship, or for success in advanced courses? Are we teaching mathematics to change our society or to establish values which will maintain our social order?

2. *What mathematical ideas, skills, attitudes, and habits can be most effectively developed at a given grade level?* The new programs have found that we *can* teach complex ideas to very young children. Now the question is what ideas *should* be taught to our pupils and at what age should they be introduced? What new topics should be introduced? What traditional topics should be dropped?

3. *How should programs be varied to provide for different levels of ability?* How do we accelerate the learning of the talented at all levels? How can small schools provide several curriculum tracks? Should enrichment include probability, calculus, or computer programing?

4. *How do we teach for transfer so that mathematical ideas will be used in solving problems?* What specific applications need to be included in the mathematics class? Are the social applications or the applications in science to be taught by some department other than the mathematics department?

5. *What degree of rigor or mathematical precision in language and logic is appropriate at various grade levels?* Should mathematical ideas be presented in simple language which, because of its simplicity, is somewhat lacking in precision? What vocabulary and symbolism is appropriate at a given grade level? How important is it to stress the basic axioms of our number system such as commutivity, associativity, or distributivity? One of the greatest dangers of the new programs is that the reorganization may go too far and confront students with concepts whose degree of abstraction exceeds the youngsters' mathematical maturity. Excessive abstraction might result in students' bewilderment and

hostility toward mathematics rather than understanding and appreciation.

6. *What emphasis should be given to computational skill?* Can this skill be attained by means other than drill? What level of competence is considered satisfactory at a given level?

7. *What is the role of the computer in the mathematics program?* Should mathematics courses teach computer programing? Should the computer be used as a tool to teach mathematical ideas and problem solving? Are computers and calculators appropriate tools for the low-ability student?

8. *How do we prepare teachers for the new programs?* How is the effectiveness of a teacher measured? What are appropriate mathematics courses for the teacher?

9. *How do we evaluate the effectiveness of a new mathematics program?* What behaviors demonstrate the attainment of objectives? What tests can be used to compare two programs each based on different content?

10. *What criteria should be used in selecting instructional materials?* What sequence of textbooks is most appropriate? Should each mathematics class have several texts and supplementary books or pamphlets? What is the role of programed texts? What is the role of concrete representation of abstract ideas?

11. *How shall the achievement of students of different ability be graded?* Should the general mathematics class as well as the accelerated class receive the entire range of grades from A to F?

12. *How are students selected for different curriculum tracks?* How can provision be made to transfer from one track to another?

In a time of change such as the present there are two extremes which can lead to difficulties. On the one hand, there is the inflexible, traditional, and conservative point of view which resists any change. On the other hand, there is the extreme liberal point of view which is ready to accept any innovation that seems popular. We need to avoid these extremes by having criteria whereby we accept, reject, or modify proposals for new programs. There are a variety of new models of school mathematics available for your school. You must make your choice. The discussions that follow will help you make a wise choice.

". . . there is a great deal more involved in the changes in school mathematics than just new subject matter. Different ways of looking at mathematics and different ways of communicating with pupils have also been involved in the recent changes in the mathematics program."

What Is the New Mathematics?

STEPHEN S. WILLOUGHBY

ONE OF the most certain methods of starting a spirited discussion, if not a violent argument, today is to mention "new mathematics." The person to whom you mention these two inflammatory words is almost certain to react with a comment such as: "They started teaching that stuff in our school last year and now none of the kids can add." Or: "They're teaching the new math in our school—my kids are excited about it and really seem to understand what it's all about, but I can't even help them with their homework anymore." Or: "How can there be a new math; isn't two plus two still four?" Or (from the expert who has read a book or taken a course on new mathematics for parents): "I don't know why they didn't teach us about sets and modular arithmetic when I was in school—maybe I'd have been good in mathematics."

The vastly divergent opinions about the new mathematics come at least partially from the fact that the term means so many different things to so many different people. One of the ironies of the new mathematics phenomenon is that many programs described as new mathematics emphasize careful definition of terms, but this term itself is not well defined.

So, what is the new mathematics? An obvious and not completely uninformative answer to this question is that new mathe-

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matics is mathematics that is new. This somewhat circular definition is informative in that it suggests that it is possible to create new mathematics, and in fact that some new mathematics has been created. Although that doesn't surprise most educated people today, there were many people ten years ago who thought that no new mathematics had been created since the time of Newton, and some who believed that the last really creative work in mathematics had been done by the ancient Greeks.

In reality, of course, mathematicians are constantly creating new mathematics, and the amount of mathematics created in any given period of time is generally greater than the amount created in any previous corresponding period of time. For example, it is an accepted fact among informed individuals that more than half of the mathematics in existence today has been created during the twentieth century.

Varied Meanings

As the layman uses the term, "new mathematics" refers to something that is going on in our schools, not to the mathematics being done by professional mathematicians; and to define "new mathematics" as it applies to the schools is a much more difficult task. The existence of mathematics that is new and the attitudes of creative mathematicians are important to the development and understanding of the new school mathematics, but there is a great deal more involved in the changes in school mathematics than just new subject matter. Different ways of looking at mathematics and different ways of communicating with pupils have also been involved in the recent changes in the mathematics program.

To make matters even more complicated, different groups have produced different "new mathematics" programs, which have incorporated new subject matter, new attitudes, and new methods which are radically different from each other. Thus, it is possible to have two programs for school mathematics, each designated as "new mathematics," which differ more from each other than from a traditional program that was in use in 1957.

In the remainder of this article, I will attempt to explain some important features that characterize many of the newer school mathematics programs, and to indicate some of the trends that appear to be occurring now.

New Subject Matter

Between the time of Newton and the beginning of the twentieth century, the major uses of mathematics were in the physical sciences. Therefore, calculus was in many respects the final goal of most secondary school mathematics. In the recent past, however, mathematics has been used widely in the social sciences, in the biological sciences, in business, and in industry. Although calculus still has an important place in the applications to these fields, there are other branches of mathematics which also have considerable importance in these new applications. Two of the most notable instances are computer science and probability and statistics.

Although there have been attempts throughout history to simplify the process of calculating, modern computers as we know them did not exist in 1945 when the second World War ended. The first high-speed electronic computer (ENIAC) was created in 1946. It could compute up to a hundred times as rapidly as mechanical computers used previously, but less than one one-thousandth as rapidly as some computers in existence today. Later development of stored programs, solid state electronics, and computer languages made the computers even more efficient.

Today, almost every major industry, business, or educational institution has a computer or has access to a computer. Many school systems use computers for scheduling and other chores. Radio and television networks use computers to help predict the results of elections. Numerous other uses are well known to every reader.

The phenomenal increase in the use of computers has many implications for the school curriculum. One of the most obvious of these is that people must be trained to work with computers. Rather than putting people out of work, what the computer really does is to make it possible to solve problems that couldn't be solved before. Particular individuals may be put out of work by the computer, but at least for the foreseeable future, more jobs will be created by computers than will be eliminated by them. Of course, the type of training required for the new jobs will be different from that required for the old jobs. At present, it appears that there are many levels of training and ability required to work with computers—from the semiskilled technician to the Ph.D. mathematician.

Computer Courses

Many school systems have begun formal or informal courses involving the computer. Some of these are designed for the most capable mathematics students, while others are for average and even below-average pupils. As of the present time, there are not many (to my knowledge, not any) programs which really prepare appropriate individuals to be computer technicians, but there are programs that are moving in that direction. More common is a course (often informal) for pupils who are good in mathematics and who would like to find out more about computers and actually to use them.

Many schools have purchased a small computer or have hooked up to a large computer through a terminal and a telephone connection. Then pupils, after a short training period, can use the computer to solve various problems both in mathematics and in other subjects. As time goes on, such programs will probably become more common, both in high school and college. There will probably be a time in the fairly near future when certain courses in the sciences and social sciences will have as a prerequisite some knowledge of computer programing.

As well as the need for people to work with computers, there is also an essential need for citizens generally to understand both the power and the limitations of computers. People must realize that a computer cannot really determine how an election is going to come out before all of the votes are cast. It can simply process in an amazingly short time information which has been put into it by human beings. But if the information or judgment supplied to it by the human beings is not good, neither will its results be good.

Other Branches of Mathematics

Although the computer revolution has had a most spectacular effect on our society, other branches of mathematics, often in conjunction with the computer, are also having very profound effects on our economy, our politics, and our scholarship. Probability and statistics have been studied for several hundred years, but have developed rapidly both in mathematical theory and in practical applications during the past few decades. The influence that statistical polls have on national policy can only be conjectured, but is certainly not insignificant.

The use of statistics in industry, medicine, education, and other fields has been increasing with leaps and bounds—usually in conjunction with computers. Today anybody who does not have at least some knowledge of statistics cannot even read the newspaper intelligently. Thus, along with some knowledge of computers, a substantial background in statistics (and the associated theory of probability) seems essential to the ordinary citizen as well as being important to many vocations.

There are many other topics which were not commonly taught in the schools a few years ago, but are presently being taught there. In some cases, the change is simply one of timing, and in others, the subject was virtually never taught in schools before. For example, geometry has been taught to school children for hundreds of years, but in this country was generally taught only in the tenth grade before the beginning of this century. During the early 1900's, some informal geometry began to creep into the junior high school grades but very little was taught in the elementary school. Today, many of the newer programs have a very substantial amount of informal geometry—both two- and three-dimensional—in the elementary grades. Considerable work with graphing and analytic geometry is also being done in earlier grades.

Sets

The new topic which has attracted the most attention is set theory. Of course, this topic like the others was taught prior to 1957. Set theory, without some of its more sophisticated twists, has been known since the late 1800's and was commonly taught in graduate courses in mathematics prior to 1950. However, even the most naive set theory is not generally taught in the school mathematics program today. Rather, what is being used is the language of sets and a few of the simpler properties and theorems of sets.

The hope of those who introduced set notation and language into the curriculum was that it would help to unify mathematics and also help children to understand the structure of mathematics. Unfortunately, there is considerable doubt that this lofty goal has been achieved. In fact, it is all too common to find a short chapter on sets at the beginning of a course in mathematics, but then to find little or no important use of that notation in the rest of the course.

It is probably true that almost everything of importance that can be said to a child about mathematics can be said without the use of the words or symbolism of sets. On the other hand, we often speak of bunches of things in mathematics. In the earliest grades, children learn to count by counting the number of objects in a collection. In graphing, the child is expected to graph a collection of points. In probability, we are interested in sets of events, etc.

It would be perfectly possible to get across all of those ideas without relying on sets, and it might even be desirable to do so in order to avoid an undue stress on the subject of sets. However, as things stand today, no major publisher would be so foolhardy as to bring out a textbook for school mathematics in which sets were not discussed at some length—there are too many people who judge a book by its coverage of sets.

Other topics which are essentially new to the school curriculum include number bases other than 10 and modular arithmetic. Originally these topics were introduced to help children understand their own (base 10) numeration system. In some cases, things have got out of hand, and different number and numeration systems are being studied for their own sake rather than as an aid to understanding the arithmetic used in our world.

There are other new topics in the school mathematics curriculum, and more new ones will be introduced. Almost certainly, one can anticipate that there will be more work with such topics as vectors, linear algebra, and numerical methods. Presumably, most of these will be introduced because they are important, worthwhile topics; it will become increasingly necessary, however, for educators to make value judgments as to which topics must be stressed heavily for which children.

New Way of Looking at Mathematics

Both the advent of high-speed electronic computers and the changing perspective mathematicians have of their subject have had profound effects on the way in which mathematics is being taught. Because computers can perform many of the routine operations, there has been more emphasis on understanding and less on actual computation in many of the newer programs. For example, in trigonometry it used to be common for pupils to spend several weeks solving triangles using logarithms and the law of tangents. Now, it is generally assumed that anybody solving a large number of triangles would find it much more efficient to use a calculator.

Logarithms are still important and still studied, but their uses are changing both in real life and in the school program. The law of tangents also is still useful, but primarily as a mathematical tool in physics (notably optics) rather than as a formula for solving certain triangles with logarithms.

Several years ago, a prominent mathematics educator suggested with a perfectly straight face that in the future each child will be issued a small pocket computer upon entry into school and will not have to be taught arithmetic—only how to use it. This is not a widely accepted point of view at present, for most people still believe that one legitimate goal of a mathematics program should be to teach children to compute accurately and with reasonable speed.

There is evidence from psychological studies that if a child understands before he practices he will be better both in computing and in knowing for which problems which operations are appropriate. Therefore, there has been much emphasis on understanding in most of the newer programs. However, when understanding has been stressed to the complete exclusion of practice (or drill), the results have generally been quite disappointing.

Radical Changes in Application

The fact that new mathematics is being created at an ever-increasing rate and new applications are being found in all fields of thought has also influenced the curriculum. About 25 years ago a large number of doctoral theses were written on the uses of mathematics in various fields, such as engineering, accounting, chemistry, etc. Apparently the inference to be drawn from these theses was that if teachers knew how mathematics is used in the real world they could do a better job of teaching it to the children. At the time this seemed to be a very reasonable conclusion.

If we look back over the past 25 years, however, we discover that the kinds and amount of mathematics used in any particular field have changed radically. A few years ago, for example, it was common for a student who was interested in science but not mathematics to go into biology. Today, advanced degrees in biology require calculus, and many specialties within biology require a great deal more mathematics. Many economists today find that they must understand game theory— a branch of mathematics which didn't really exist 25 years ago. Physicians are using topology, analysis, computers, and other fields of mathematics.

Many professions are using mathematics today which was virtually unknown 25 years ago. This suggests that if a child learns the mathematics appropriate for his chosen profession at the age of 15, his knowledge of mathematics will be obsolete by the time he is 40. Clearly, it is more important to teach people how to learn and use new mathematics than to teach them the mathematics that is presently being used in their chosen profession. This consideration, along with some information from learning theory, has encouraged many teachers and educators to emphasize understanding and creativity on the part of the pupils rather than simple rote-learning.

Again, a word of caution is necessary. Most of the mathematics presently—and previously—in the curriculum is of proved value. It would be foolhardy simply to ignore all of this mathematical knowledge and to ask that pupils simply understand some mathematics, however irrelevant.

Structure

An emphasis on mathematical structure is a means some programs have used to help pupils understand the mathematics they are learning. Thus, rather than learning individual mathematical facts in a disconnected manner, the child is expected to get a feeling for the entire system. Prior to 1800, an axiom system was generally thought to be a set of truths about some portion of reality from which theorems could be derived. With the advent of non-Euclidean geometry and algebraic structures, our concept of what an axiom is has changed.

Today, an axiom is thought of as an assumption which in conjunction with other assumptions will produce a reasonably good mathematical model of some portion of reality. There is no reason why certain assumptions cannot be changed if it is convenient to do so. By changing one or more assumptions, we get a new system which may approximate the given portion of reality more closely, or the change may produce an interesting mathematical structure which has no immediate application to reality. There have been many cases in which abstract mathematical systems were developed that appeared to have little connection with reality but which were later shown to be of great practical significance.

As the study of structure applies to the elementary and secondary school, it is commonly thought that children should have some experience in creating their own axioms to describe a situation. They should have experience stating their own theorems and trying to prove them. They should have the experience of conjecturing false as well as true theorems, and they should even be encouraged to try different axiomatic systems to describe the same situation.

This is a somewhat different situation from the one in which axioms are given to the class by the textbook or the teacher, then true theorems are presented, and pupils are asked only to produce valid proofs for theorems they know to be true, by using axioms they believe to be unquestionable truths. Notice that this new point of view would encourage deductive reasoning and study of axiomatic structures in fields other than just geometry. Indeed, one of the notable differences in the newer programs is the emphasis placed on deductive reasoning in algebra, arithmetic, probability, and other branches of mathematics as well as in geometry.

Functions

There are other changes in the curriculum caused by a different way of looking at mathematics. One example of this is the increased emphasis on functions. Functional analysis has played an increasingly important role in mathematics, and it is therefore appropriate that functions are studied more thoroughly in algebra courses and that trigonometry be approached through circular functions in the eleventh or twelfth grade rather than as the study of right triangles. The circular function approach to trigonometry is both more efficient and more in keeping with the uses of trigonometry in higher mathematics and science.

In recent years there has also been an increased emphasis on the foundations of mathematics. It has been discovered that a lack of care can lead mathematicians into embarrassing paradoxes. Because of this, many of the school programs have required a great amount of rigor of the pupils. For example, it has long been known that there is a difference between an object and the name for that object. (Children have long known puzzles such as the following: Show that half of 18 is 10. *Answer:* Take either the top half or the bottom half.)

Although it is perfectly correct to point out that a numeral is a name for a number and that a number is really an idea not a symbol on a piece of paper, there is some doubt in the minds of many educators that a consistent and careful distinction between objects and their names really serves a useful purpose. Unfortunately, there are some teachers, and even authors, who have given great emphasis to this distinction but have actually done it incorrectly—certainly this is not good.

Other examples of great rigor can be found in many of the newer programs, but it is my opinion that such rigor should generally come as a reaction to a problem that the students understand, rather than being impressed upon the students because some mathematician has discovered the rigor is necessary to make the foundations of the subject meet present-day criteria. Thus, if pupils are having trouble because they are confusing names and objects, the distinction should be made—otherwise it is superfluous. In a similar way, if children are worried about whether one of three points on a line is between the other two, some axioms of betweenness are appropriate; otherwise they may do more harm than good in terms of ultimate understanding.

Methods

In many respects, the teaching methods of the modern mathematics programs are inherent in the subject matter and attitudes already discussed. Most of the newer programs emphasize pupil activity of one sort or another—often this is described as the "discovery" approach. Even before the recent revolution, there were prominent mathematics educators who advocated more emphasis on understanding and children's thoughts. In fact, as long ago as 1821, Warren Colburn began publishing a series of mathematics textbooks based on the theories of Pestalozzi, which encouraged pupil activity and thought. Although at several periods in our history the teaching of mathematics has gone in other directions (the stimulus-response era of the 1920's and 30's is an example), many good teachers have always taught for understanding.

The importance of applications of mathematics leads one to the belief that mathematics should be closely tied to the real world when it is taught. This has encouraged many physical models in the classroom and various other experimental materials that can lead to better understanding of mathematics. When NDEA funds became available for such materials, there was a substantial increase in the number of them available in schools.

Another change in methods of teaching which may become very important is an outgrowth of two different trends in psychology. These are the increasing awareness of individual differences, and the continuing influence of the behaviorist, stimulus-response school. Teaching machines and computer-assisted instruction are being considered as serious adjuncts to the teacher in many places. Most of the fabulous claims that heralded the entry of these devices into the educational world have since been shown to be exaggerations, but there is still reason to hope that both programed learning and computers may assist the teacher in many ways to do a better job.

It is worth noting, however, that if teaching machines and computers are being thought of as relatively inexpensive ways of alleviating the teacher shortage, they will almost certainly not achieve the desired goal. Furthermore, in mathematics anyway, there is considerable evidence to support the contention that the teacher shortage as we have known it for the past 25 years will be considerably eased within five years.

In Brief

To summarize, several important innovations characterize the new mathematics programs for schools. In my opinion, among the most significant of these are:

1. *New subject matter.* In particular, the schools are beginning to give more work with computers, probability and statistics, linear algebra, and substantial work with inequalities.

2. *Changing placement of subject matter.* Examples are the study of informal geometry and algebra in the elementary school, the moving of solid geometry and some analytic geometry into the tenth grade, and increased use of the "spiral approach" in which a subject is studied several times with ever-increasing maturity.

3. *Modification of subject matter.* Increased emphasis on functions in the study of all branches of mathematics, especially trigonometry and algebra, is an important example of modifying the

subject matter. Another important example of this is the increased use of a deductive approach to branches of mathematics other than geometry.

4. *Increased understanding.* Evidence suggests that children who understand a process before practicing it learn it more efficiently and can use it more effectively. Therefore, understanding why certain algorithms work has been much emphasized both in elementary and secondary schools. It is also hoped that increased work with mathematical structures and their relation to reality may help children to understand and appreciate mathematics.

5. *Learning how to learn.* Perhaps the most important aspect of a good mathematics education program is to teach children how to learn and how to be creative. Those who can learn new mathematics when needed and can be creative about the way in which they apply it will always find a use for their talents. Unfortunately, this is also one of the most difficult things to teach.

There are other characteristics of many of the newer programs, and there is a great variety of opinion as to whether certain characteristics are good or bad. Those administrators, teachers, and parents who are in positions of influence with regard to which programs are used should be sure that they understand why certain changes are being made and what improvements can be expected from the changes. They should not accept change simply because it is the current fad, nor should they assume it is successful simply because it is new.

While we speak of the *revolution in the mathematics curriculum, it is evident from the descriptions presented by Professor Nichols that this revolution is not the work of a single general, nor even of a clique of the highly placed. From all quarters of the academic community, forces both great and small are assaulting the traditional barricades that have blocked off access to real mathematical understanding.*

The Many Forms of Revolution

EUGENE D. NICHOLS

IN HIS contribution to the eight Regional Orientation Conferences in Mathematics conducted in 1960 by the National Council of Teachers of Mathematics and supported financially by the National Science Foundation, G. Baley Price stated that "the changes in mathematics in progress at the present time are so extensive, so far-reaching in their implications, and so profound that they can be described only as a revolution."[22]* He attributed these changes to three causes:

1. the tremendous advances made by mathematical research
2. the automation revolution brought about by the introduction of machines that control machines
3. the introduction of high-speed automatic digital computing machines.

Since that time, changes in each of these areas have accelerated. More significantly, these changes have begun to affect the school curriculum at both the elementary and secondary levels.

* Numbers in brackets refer to sources listed alphabetically at the end of this article.

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As is usually the case with curricular innovations, they begin with a small group of "idea" people, who, because of their intimate knowledge of the subject, are in a position to propose changes in the content of the curriculum. Obsolete topics are discarded and topics which have taken on new significance are added. As a result, a new curriculum emerges, frequently drastically different from that of the past.

The first impressive signs of serious dissatisfaction with the traditional mathematics curriculum preceded the launching of Sputnik in 1957 by about six years. At the University of Illinois a group under the the direction of Max Beberman concluded that the traditional subject matter of mathematics and the mode of its teaching at the secondary level needed a complete re-examination. The group proceeded to develop a new curriculum, which was based on three major theses:

"1. that a consistent exposition of high school mathematics is possible;
2. that high school students are greatly interested in ideas; and
3. that acquiring manipulative skill and understanding basic concepts are complementary activities." [32]

Since that time a number of other groups have undertaken the task of contributing to the revision of the school mathematics curriculum. This revision has become an ongoing process, with experimental groups pouring out new ideas and materials and with teachers and students testing them for teachability and providing the feedback for subsequent revisions.

The process of revising the curriculum is most complex. While in the past there seemed to be almost universal agreement as to what should be taught at a particular grade level, there is no longer such an agreement. Field tests of topics once assumed too advanced for a particular level have demonstrated the assumption to be false. Not only were students able to master the topics, but they pursued their learning with a degree of enthusiasm and excitement seldom seen previously in the classroom. Such experiences have suggested a need for new guidelines on grade placement of topics. The construction of such guidelines is one of the many tasks still to be undertaken.

Experimentation with new mathematical content and with new strategies of teaching has taken many forms. A variety of new topics has been introduced under the guidance and with direct participation of active mathematicians. The experimentation with new strategies of teaching—involving such technological devices as video-tape recorders, electronic computer systems, and films—has been carried on by classroom teachers and mathematics educators.

Current Curricular Projects

The rest of this article is devoted to brief descriptions of a number of curricular projects. There is not enough space here to do justice to all aspects of each project, nor is it possible to include all ongoing projects. This is merely a sampling which hopefully will enable the reader to perceive some trends and to appreciate the complexity of the entire development.*

It is not possible to present here all of the projects which have relevance to the school mathematics curriculum. It is hoped, however, that a sufficient number and variety of projects are described to give the reader a balanced picture of what has been happening and is happening on the school mathematics curriculum scene.

University of Illinois Committee on School Mathematics

Since its founding in 1951, UICSM has directed its efforts toward secondary school mathematics curriculum revision and the development of new teaching techniques. In its first decade UICSM, under the direction of Max Beberman, produced an 11-unit sequential curriculum for grades 9 through 12, principally for college-capable students. Originally duplicated in loose-leaf notebooks, the series is now available in four hard-bound editions.

The teaching innovations introduced by UICSM come under the heading of the "discovery method," the UICSM technique falling in the specific category called the "nonverbal awareness method." Basically students are taught by leading them into making their own discoveries about mathematical concepts, but are not forced to verbalize the concepts they have found. This places the emphasis on the student doing mathematics in contrast to being told how to do mathematics by a set of formulas. Those who worked on this project report great success with the method, not only in the teaching but in providing motivation.

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Concurrent with the development of the texts, UICSM has run other projects such as teacher in-service and summer institutes primarily concerned with the preparation of teachers qualified to use UICSM materials. A programed instruction project was also conducted.

To accompany its "discovery method" innovations, UICSM has produced many films showing actual classroom presentation of its materials. These films were not produced because of their mathematical content, but to demonstrate the discovery method.

Since 1962 UICSM has moved into three new areas of curriculum development: an arithmetic of fractions sequence for seventh- and eighth-grade low-achievers, a junior high informal geometry course, and a vector geometry course. Once again teacher training institutes accompany the projects.

The fractions course employs such terms as "stretchers," "shrinkers," and "hookups," giving the students extensive work with fractions before they are ever told that these things they have been studying are in reality just fractions.

The informal geometry course makes extensive use of translations, rotations, and reflections to introduce congruence geometry. Originally designed for low achievers, this course can serve as supplemental material at all ability levels. Vector geometry is intended for high-ability secondary students and for teacher education at both the undergraduate and graduate levels.

Throughout all its materials UICSM has maintained the position that the language employed be as unambiguous as possible and that emphasis be given to consistency. All materials have undergone classroom testing in participating schools and school systems. Most of these schools, but not all, have used the UICSM sequence under the direct supervision of the project center.[1, 2, 8, 9, 20, 23, 29, 31, 33]

Madison Project

The Madison Project was founded in 1957 at Syracuse University to promote more effective teaching of mathematics and to develop a supplemental program in mathematics for grades K-12. An auxiliary center was started at Webster College, St. Louis, Missouri, and the project now makes its main headquarters there. The name is taken from Madison Junior High School in Syracuse, where the project's materials were first introduced. Its director is Robert B. Davis.

The Madison Project is not concerned with an overhaul of existing mathematics curricula, but has been writing what is termed a supplemental program designed to "give life" to school mathematics; that is, to stimulate children to be creative in mathematics and to develop enthusiasm for it.

Part of this aim is accomplished by providing stimulating classroom experiences, or what the project calls "mathematical laboratories." The "discovery method" is very much a part of these mathematical laboratories. Films and video tapes of Madison Project classes in action have been made and are available to the public. The films are intended for use by teachers and prospective teachers in their preparation for effective ways of introducing mathematical topics.

The supplemental program—which supplements the standard, everyday curriculum—deals with two levels: (1) novel ideas unfamiliar to the student but which will whet his mathematical appetite, and (2) more familiar topics designed to increase the student's skills in basic concepts.

Two texts are commercially available—*Discovery in Mathematics* and *Explorations in Mathematics*.[12,13] The first is suitable for grades 4 to 8 and the latter for grades 6 to 9; both come with teacher guides. Topics for the 4 to 8 levels include axiomatic algebra, coordinate geometry, and applications to science. At the 6 to 9 level, the topics are statistics, logic, matrix algebra, and applications to physics.

Teacher training in the use of Madison Project materials is provided at Syracuse University and Webster College. Besides the large number of films, the project holds a number of special workshops around the country to acquaint teachers with its materials. Two in-service packaged courses have been produced which present all aspects of the project—films, written materials, and laboratory equipment.

Many schools from the New York, St. Louis, and Washington, D.C., areas are using the Madison Project materials. Studies of the effectiveness of the project are conducted right in the classroom; in addition, the project is following students through several years of schooling to record their progress.

University of Maryland Mathematics Project

Junior high school textbooks and textbooks for elementary teachers have been the main contributions of UMMaP in its effort to improve the teaching of mathematics. Begun in 1957, the project has been supported by the U.S. Office of Education; John R. Mayor was its director.

At the time of its founding, UMMaP was interested in the development of experimental courses in mathematics at the junior high school level. These courses have been produced, and the project has now turned its attention to the mathematics education of elementary teachers.

Two texts for junior high school have been written and are available commercially—*Mathematics for the Junior High School: First Book* and *Mathematics for the Junior High School: Second Book*.[34] Junior high school teachers from the university area (Washington-Baltimore) met for three years with writing teams from the University of Maryland; and the cooperative efforts of both produced many teaching units, which after two years of classroom testing and revision were finally organized as the two texts.

Language and structure of mathematics are stressed in the UMMaP-produced texts, and distinction is made between mathematical symbols and the mathematical concepts the symbols represent. There is an emphasis on number systems, and the sequential development of more difficult number systems from the less difficult. Some of the less standard topics included in the books are statistics, probability, logic, and trigonometry.

The texts written for use by elementary teachers are *Mathematics for Elementary Teachers: Book I* and *Mathematics for Elementary Teachers: Book II*. These are geared for the college classroom training of prospective teachers and have been adopted by the University of Maryland and several other colleges. Book I deals with number systems of ordinary arithmetic and Book II with geometry, particularly measurement. Deductive procedures are employed in the texts.

Current emphasis of UMMaP is on the development of an in-service course for elementary teachers which will include a great deal of learning psychology or theory. It is hoped this course will serve as a model for other institutions initiating a similar course. UMMaP will be conducting research in the learning of mathematics in conjunction with the development of this course.

Boston College Mathematics Institute

Improvement of the content and the teaching of college and secondary school mathematics is the objective of this institute, which takes a three-way approach toward achieving its aim: (1) written materials for college and secondary school level, (2) course content institutes for teachers and experimentation with new course material, and (3) a computer program for high schools.

Work on the written materials and the institutes has been in progress since 1957, while the computer program is a 1963 addition. Stanley J. Bezuszka, S.J., chairman of the mathematics department, directs the Boston College Mathematics Institute.

Written materials have centered on texts for junior high school and basic "modern math" correspondence course texts for teachers. Institute director Stanley Bezuszka, with the help of his staff, has authored all of the books. The junior high texts, part of the *Contemporary Progress in Mathematics* series, were written primarily for use in classrooms where the teacher has had little background in updated mathematics.[5] Topics include the integers, rational numbers, euclidean geometry, bases, set theory, puzzles, and more. The books were designed so that the student could effectively read and learn by himself. A seventh-grade text aimed at the below-average student is also available.

Self-study texts, *Cooperative Unit Study Program*, Courses 1 and 2, were written for use by mathematics teachers with minimal "modern math" backgrounds.[6] Although they are used primarily in correspondence courses offered by the institute, once the material in the texts has been mastered, it can be used as reference material for classroom lectures. Similar self-study books are planned for the elementary teacher.

The Institute has also constructed various drill machines and learning devices. Other projects involve writing a calculus text, a text in modern mathematics for parents, and a computer-oriented mathematics text for high school.

In the area of course-content institutes, the Boston College Mathematics Institute has offered National Science Foundation academic year institutes and summer institutes. These programs can lead to the nonresearch master of arts degree in mathematics. Since 1957 they have been expressly for the training of secondary school mathematics teachers, with the summer institutes designed chiefly for those teachers with inadequate prior training. Demonstration classes, composed of students from the Boston area, have been an integral part of all institutes. In addition the Institute has held workshops in several parts of the country, such as California and Hawaii.

Experimentation in high school computer-programing courses is the third item in the Institute's objectives. Students are taught the elements of computer operation and programing as applied to mathematical problems. This experimentation has taken place during summer sessions and regular academic years at Boston College's computing center and at several local high schools. Student participation has been voluntary, and the experimental classes have handled students of all secondary grade levels, including one high-ability sixth-grader. As of now, the Institute concludes that the elements of computer science can be successfully taught in the high school years, while actual computer operation appears to be a waste of time. Eventually a decision will be made as to whether this is best taught in one year or throughout a four-year sequence, bringing computer techniques into math classes as they are needed. A text is being prepared.

Brighter students seem to enjoy working out problems beyond those presented in the classroom, whereas average students are satisfied with relating their computer work with their present level of mathematical skill. Training of teachers in this area is also anticipated.

It is the belief of the Boston College Mathematics Institute that no sooner will the nation's math teachers have adjusted to the change to contemporary curriculum than they will be expected to adopt this new trend—computer-oriented mathematics.[3, 4, 7]

School Mathematics Study Group

Founded in 1958 on guidelines established at a meeting sponsored by the National Science Foundation at the Massachusetts Institute of Technology in February 1958, SMSG has had as its major thrust the preparation of sample textbooks to lead the way into the modern mathematics curriculum. Through the years, over 60 such texts have been written by SMSG teams or individuals, as well as supplemental texts, monographs, conference reports, programed learning materials, and Spanish translations.

The emphasis has been directed toward a revamped secondary school curriculum, but current projects are aimed at the development of a comprehensive curriculum from kindergarten through grade 12. Initially SMSG wrote its texts for the average to above-average student, but several later projects and writing sessions have produced materials that encompass all ability levels. Writing teams have included school teachers of mathematics, mathematics educators, psychologists, and mathematicians.

Director of SMSG is E. G. Begle. The project was originally established at Yale University, but was moved in 1961 to Stanford University. The National Science Foundation has funded the project.

In the area of teacher-training, SMSG has written some texts principally for in-service institutes and summer institutes, plus teacher's manuals to accompany all the major textbooks for students. Their principal objective in this area has been the immediate preparation of teachers able to teach the SMSG curriculum.

The purpose of the SMSG texts has always been to provide a model for commercial writers, although the SMSG texts have found widespread classroom adoption. Before the books were placed in teachers' hands, they received classroom evaluation and revision by the writing teams and editors. Once commercial texts in the modern vein became available, SMSG promised to get out of the publishing business. A 1966 statement by Mr. Begle attests to this: "Since, on one hand, an increasing number of improved textbooks are available and the sale of SMSG texts is decreasing; but since, on the other hand, the SMSG texts are still useful in helping teachers bridge the gap between a traditional program and a modern one, it has been decided that these texts will be kept in print as long as there is any demand for them and will be allowed to die a natural death when the demand disappears." [24, p. 5]

Available texts from SMSG range from the elementary level to calculus. The geometry series and the junior high series have been very popular during the years of transition from the traditional to modern curricula.

Current projects include a program for low achievers in junior high school, a broad study of gifted students, continuation of a new sequence for secondary schools, and conclusion of a five-year study of comparative mathematical achievement. The new sequence will primarily be the classroom revision of a seventh-grade text and the writing of an eighth-grade text. SMSG believes the junior high school years mark a transition period where mathematics should be moving from applications in the real world into abstractions. The new junior high school texts will reflect this philosophy by mixing the two approaches to mathematics.

SMSG has been following the progress of students training with SMSG materials for four years now and this year is the last of their five-year study. Results will be processed as the data come in. In other research SMSG hopes to "ascertain the differential effects in student achievement of different kinds of textbooks, including both modern and conventional ones. The second [research project] is designed to ascertain the differential effects on student achievement of individual differences in students, teachers, schools and communities."[25, p. 8] With this type of information SMSG will tackle such problems as how to assign students to an eighth-grade accelerated algebra class on the basis of seventh-grade information.

The interested public is kept abreast of SMSG developments by means of newsletters, which are available upon request from the project headquarters.

University of Illinois Arithmetic Project

This project's main concern is the development of selected topics in mathematics for grades K-6. A systematic curriculum revision is not the purpose of the Arithmetic Project, rather it is the production of "here-is-something-to-try" ideas for the classroom. The project staff, which David A. Page heads, believes curriculum development is not really possible until more alternatives exist; it is their aim to produce some of these alternatives.

Founded at the University of Illinois in 1958, the project moved in 1963 to Educational Services Inc. ESI subsequently became the Education Development Center with headquarters in Newton, Mass.

It is the philosophy of the project that young people should find adventure in their school mathematics, although these adventures will require and deserve hard work. "Students who are not to continue a formal study of mathematics deserve a taste of the subject that is at least as appealing."[30] In other words, the project is attempting to motivate the study of mathematics through topics of interest and challenge. It has been found that students will go through vast amounts of computation to solve problems which interest them and as a consequence their computational skills improve with the use of project materials.

Several booklets have been prepared on such topics as estimation, number-line and number-plane jumping rules, and maneuvers on "lattices."

To implement their ideas, the project has been teaching these topics in many of the local schools in the vicinity of the project center. Teachers have been training at in-service institutes at the local schools or at the Massachusetts state colleges. Presently the Arithmetic Project is writing an in-service course which can be taken by participating teachers without expert mathematical guidance.

Films of demonstration classes have been made, and the project is working towards a film series which will depict the presentation of an entire course.

Greater Cleveland Mathematics Program

The aim of GCMP has been to develop a comprehensive, sequential mathematics program for all those children in kindergarten through grade 12, a program which is both mathematically correct and pedagogically sound.[18, p. 1] George S. Cunningham is the program director.

GCMP was begun in 1959 when the advisory committee of the Educational Research Council, a nonprofit organization whose purpose is to improve elementary and secondary education, asked the Council to direct its efforts toward improving the mathematics curriculum. The materials produced by the group are being used in more than 30 participating school systems.

An elementary program has been completed and commercially published, while a junior high program is currently under development and testing. The senior high school program has been planned, and a senior high geometry text is already available and in use.

The GCMP curriculum is based on the needs of the participating systems and the recommendations of the National Council of Teachers of Mathematics, Mathematical Association of America, and the Commission on Mathematics of the College Entrance Examination Board. The central theme of the elementary GCMP is a thorough investigation of the set of rational numbers and its principal subsets—the set of whole numbers, the set of integers, and the set of fractional numbers. The geometrical concepts in the elementary program include points, lines, planes, and their interrelationships. Throughout the program there is an emphasis on problem solving, beginning with "trial and success" techniques. [19] The elementary school materials have undergone three major revisions since 1959. The need for these revisions was indicated by GCMP's continual testing program and feedback from the participating teachers.

At the secondary level the GCMP calls for a tracking system with three levels of ability, planned to meet the needs of all students. The student may spend all three junior high school years in pre-algebra or may advance as far as Algebra I and II, plus some geometry. The already published geometry text covers the postulational method through informal deductive geometry, basic geometry of the plane and space, euclidean geometry, and non-euclidean geometry

To provide teachers with the background necessary to be effective in working with the GCMP materials, the following services have been made available: in-service teacher-training sessions, TV teacher-training programs, teacher-training films, teacher guides to pupil material at each grade level, and teacher textbooks.[18, p. 3] The project also makes use of the team-teaching approach.

GCMP reports that testing and feedback show student attitude improved, computational efficiency improved for all including the below-average students, and quality of teaching improved.

Minnemast

Started in May 1961, the Minnemast Project is attempting to develop a coordinated math-science curriculum for kindergarten through grade six.[21] Its heaviest emphasis to date has been on mathematics.

Minnemast, which is short for Minnesota Mathematics and Science Teaching Project, is a product of the University of Minnesota under a National Science Foundation grant. James H. Werntz of the University of Minnesota's physics department is the project director, and Paul C. Rosenbloom of Teachers College, Columbia University, is the project mathematics consultant.

Basically, the project has come up with an entirely new curriculum for the primary grades. Children are taught such concepts as numeration and symmetry through discovery, observation, and investigation. Geometry, for example, is introduced at the second-grade level. Games, stories such as "Squareville" which introduces the Cartesian coordinate system, and other teaching devices which serve as motivational material are widely used. Discovery of patterns is emphasized.

Current work in mathematics is based on the philosophy of the Cambridge Conference on School Mathematics held at Harvard University in 1963. Science and mathematics have been integrated for kindergarten and grade one, but the project's ultimate goal is to combine the two disciplines for all the elementary grades. Fundamental to the project's goals are four competencies to be acquired by boys and girls: (1) to know how a scientist acquires knowledge; (2) to observe, classify, measure, generalize, predict, and test predictions; (3) to recognize problems and discover answers; and (4) to build a coherent structure of scientific knowledge.

Minnemast utilizes 10 experimental centers at colleges around the country to field test its materials. The Minnemast staff realizes the teacher of today's average elementary school is not properly trained to present the Minnemast curriculum; consequently, many of the centers are involved with in-service education of teachers.

Also under development at Minnemath (the name given to the project center) is a college-level geometry "package." It is aimed at future high school mathematics teachers. The package includes filmed lessons, programed study units, and comprehensive texts and problem materials. Completed or under development are 11 texts, each covering a different topic in geometry. Seymour Schuster, Minnemath director, is in charge of the geometry project.

As with Minnemast, the geometry package is being tested in various schools. Closely tied with the programed study units is another Minnemath project—a programed correspondence course in geometry for in-service education of secondary teachers. The geometry package is intended to meet the recommendations of the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America. It is hoped it will upgrade geometry instruction in high schools.

Minnesota National Laboratory Evaluation Projects

The Minnesota National Laboratory, a branch of the state of Minnesota's Department of Education, is involved in two projects. One is entitled "Evaluation of Secondary Mathematics Curricula" and the other "Effects of Modern and Conventional Mathematics Curricula on Pupil Attitudes, Interests and Perception of Proficiency."

The object of the first is to determine the relative effectiveness of four experimental mathematics programs as indicated by pupil achievement. The four programs under evaluation are the School Mathematics Study Group (SMSG); the Ball State Teachers College (Ball State); the University of Illinois Committee on School Mathematics (UICSM); and the University of Maryland Mathematics Project (UMMaP). Teachers from Minnesota, Iowa, Wisconsin, North Dakota, and South Dakota volunteered for the project, which involved approximately 1,500 class units in grades 7-12.

Students were tested prior to entering the program and at various stages throughout the program. In addition, control groups of conventional classes were involved. Through observation and classification, teacher instructional characteristics that could reflect differences in the experimental materials or that might have had an independent effect on achievement were investigated. According to a summary report: "Analyses to date indicate reliable achievement differences favoring some experimental programs at some grade levels. However, these differences tend to be quite small especially in comparison with the differences attributable to the pupils' initial level of proficiency." [14] Other relevant analyses such as sex differences, class size, and school characteristics are also slated to be carried out.

The second project is closely allied to the first and is being run primarily on ninth-grade students already participating in the evaluation project. Thus the same four experimental programs are involved. The project report states: "The most general purpose of this project is to determine the effects of several recently developed experimental programs in mathematics on the attitudes and interests pupils develop toward mathematics as a school subject and as an extracurricular activity. . . . The project is also concerned with determining the more specific pupil perception and judgment factors and pupil characteristics relevant to instruction that are related to general attitudes and interests developed toward mathematics."[14]

Under study are such factors as pupil perception of mathematical knowledge, learning difficulty, motivation through subject matter, motivation through grades, and perceived utility of mathematical knowledge. Requests for activity bulletins were used for an index of pupil interest in mathematics as an extracurricular activity. Measures of teacher attitude toward and judgments about the experimental materials were also obtained.

Stanford Program in Computer-Assisted Instruction

This broad-range program in computer-based instructional systems places a great deal of emphasis on mathematics teaching, probably because of the mathematical interests of the program director, Patrick Suppes. Elementary mathematics has been the chief concern in this expanding program, which also includes one project at the ninth-grade level.

Headquarters is the Institute for Mathematical Studies in the Social Sciences at Stanford University. Begun in 1963 with National Science Foundation, U.S. Office of Education, and Carnegie Corporation grants, the Stanford program attempts to utilize computers at three teaching levels—drill and practice, tutorial, and dialogue. According to Mr. Suppes, "One of the most exciting aspects of computer-based education is the opportunity it offers for tailoring instruction to the individual child's needs."[28] An explanation of the three teaching "systems" will make this individualization clear.

Drill and practice is a supplement to the regular teacher-taught curriculum. The child sits at a station where he is presented drill problems by the computer. The child may respond, depending on the computer system, to each problem by means of a cathode ray "pen" or a typewriter keyboard. His answers are checked by the computer, correct or incorrect response indicated, and a tally kept of his responses. The student starts at an intermediate level of difficulty each day and, depending on his success, the computer adjusts the level of difficulty up or down. At the end of the day the teacher can get an immediate summary from the computer of each child's responses, and thus identify areas where more classroom instruction is needed.

The tutorial system is a complete instructional sequence in a given subject. This type of system lends itself well to what is commonly called "programed teaching," but differs in that the computers are programed to give hints or other aids and to allow more latitude in instruction. In addition Stanford uses sound systems whereby the computer "talks" to the child, presenting the day's lesson.

The third level of teaching systems, called the "dialogue system," is strictly a thing of the future, because it calls for speech recognition by the computer. In this way the computer and student could interact much as students and teachers do at present.

Mathematics courses under experimentation have been a logic course and modern algebra. Students have ranged from gifted to below average. Currently a remedial arithmetic course is being offered in an area junior high school. Participating schools have been primarily from the Stanford area, but schools in Kentucky and Mississippi have been linked by teletype to the Stanford-based computer.

Mathematical concepts covered at the first-grade level in the computer-assisted projects have included telling time, counting coins, measuring segments and polygons, taking one-half an object, concave figures, addition and subtraction, and recognition of sizes and positions. Bright second-graders have been given coordinate geometry, numeration systems, negative numbers, linear equations, three-dimensional geometry, and counting in different bases. At the more advanced elementary level, axiom systems, associative and commutative laws, and mathematical proofs have been introduced.

There are many considerations in a computer-assisted mathematics program, both advantageous and disadvantageous, but despite the relative newness of this method, the Stanford program is achieving success in the participating schools.

Secondary School Mathematics Curriculum Improvement Study

This project, based at Columbia University, restructures the traditional curriculum of grades 7 through 12 by introducing experimental texts for each grade level. The principal objectives of the Study are to formulate, experiment with, and evaluate a secondary school mathematics program which will take capable students well into what is currently collegiate instruction in mathematics.[26] The primary means of attaining this goal is a special ordering of mathematical topics which will give an efficient and unified introduction to each concept in the revised curriculum.

Under the direction of Howard F. Fehr, SSMCIS is being conducted through Teachers College of Columbia University. Each of the new SSMCIS texts will receive classroom testing; the seventh-grade book was tested during the 1966-67 academic year and the eighth-grade book is being used during 1967-68. Test schools are primarily in the New York City area, and test classes total 350 students.

The curriculum restructuring consists essentially of introducing advanced topics at earlier levels. As an example, the topics for the seventh grade, in order of presentation, are: Finite Number Systems; Operations and Sets; Mappings; Integers (Z, +); Probability and Statistics; Integers (Z, +, −, •); Lattice Points in a Plane; Sets and Relations; Transformation of the Plane; Segments, Angles, Isometries; Elementary Number Theory; the Rational Numbers; Applications of Rational Numbers; and Flow Charting Algorisms.[27]

As part of the project, SSMCIS is attempting to find out if teachers can be educated to handle this advanced curriculum. Twenty teachers were given extensive summer training before the introduction of the seventh-grade text. The eighth-grade program is now undergoing classroom revision. Topics for grades 9-12 have been picked, but the ordering for each year has not yet been established. A list of these topics is available from the SSMCIS center.

A limited number of copies of Course I (seventh grade) are available to educators this year, and will be available in 1968-69 to school districts that wish to have the course taught and that have qualified teachers. Course II will become available in 1969-70 on the same basis.

Two research studies, the learning of the concept of function and verbal comunication in mathematics education, were conducted concurrently with the experimental teaching of the seventh-grade text and are near completion. Two others, the learning of the concept of real number and the relation of teacher knowledge of mathematics to class achievement, are being prepared.

SSMCIS is the outgrowth of a June 1966 conference of American and European mathematicians and mathematics educators. At the conference the group laid the groundwork for the project by writing a topical syllabus and flow chart of mathematical concepts for the secondary grades. The actual textbook writing has been done by a team of writers with extensive secondary school experience.

Comprehensive School Mathematics Project

Totally individualized instruction is the ultimate goal of this project. Under the direction of Burt Kaufman, Jack Kelley, and Roger Robinson, all of Southern Illinois University, the project is supported by SIU and the Central Midwestern Regional Educational Laboratory.

CSMP, based on the recommendations of the Cambridge Conference on School Mathematics at Harvard University in 1963, is a continuation of the Nova Project, started by Burt Kaufman in 1963 at Nova High School, Ft. Lauderdale, Florida. During 1967-68 three levels of development are being undertaken: (1) individualized curriculum development through the writing of "activity packages"; (2) continued development of the program for gifted secondary school students; (3) development of an elementary school program.[10]

By individualized instruction CSMP means activity packages which take each student as far into mathematics as his abilities allow. Traditional classrooms will become unnecessary, and study carrels accommodating several students are envisioned. The latter is a breakaway from the standard idea of one student per carrel, because the project staff believes small-group interaction is necessary for effective learning. Promotion will be based on successful completion of units rather than courses. The project also plans to use team teaching and audiovisual aids in the development of the activity packages. Computer-assisted instruction is contemplated for the future.

On the full-time CSMP staff are both a systems analyst and an educational researcher, who will evaluate the project at all levels of development. Not only will the project be tested against its own goals, but its effectiveness will be checked against other curricula across the country. Initially, the elementary program will derive its curriculum from noncommercial sources such as the Madison Project, Minnemast, University of Illinois Arithmetic Project, African Mathematics Program, SMSG, UMMaP, UICSM, and others. At the secondary level present plans call for a mixture of experimental materials and some commercially published mathematics texts.

Classroom instruction under CSMP is presently based on a track system, which roughly divides students into low, average, and high ability, but total individualization will eventually go far beyond tracking.

The 50-man Advisory Board, chaired by Robert Davis of Syracuse University and the Madison Project, includes many well-known mathematicians, psychologists, educators, and mathematics educators from both the United States and Europe. Several subcommittees have been or are being formed to advise each segment of CSMP.

Des Moines Public Schools Experimental Project in General Mathematics

Designed for the under-achiever or low-achiever, this project is in its fourth year of development in eight Des Moines, Iowa, junior high schools.

In order to incorporate Swiss psychologist Jean Piaget's philosophy of "active" learning, the project makes use of printing calculators, typewriters, copy machines, overhead projectors, controlled readers, and filmstrips, film projectors, opaque projectors, tape recorders, and other aids. Semiclassical music is played on the radio during activity periods. A booklet of supplemental materials has been written, and is entitled LAMP (Low-Achiever Motivational Program).

Essentially, the program is a laboratory approach to mathematics. Several "real life problems" from Des Moines business and industry are also incorporated. It is a teacher-centered program with a goal of leading the low-achiever from apathy to enthusiasm for mathematics. Some early success has been reported. [15]

Computer Mathematics Project

The Massachusetts Board of Education is testing the use of the time-shared computer as a teaching aid in mathematics. Under the direction of Jesse O. Richardson and supported by the U.S. Office of Education, it is hoped through this project to develop a supplemental mathematics curriculum which will stimulate the study of mathematics as well as teach computer techniques.

Several schools in the New England area are involved in the project. Materials are being developed for grades 6-12. Basically, the project involves teletypewriter terminals in the participating schools which are hooked up to the central computer. A student in any participating school can sit down at the terminal and communicate instantly with the computer.

Several summer workshops have been held to acquaint teachers with computer techniques, and some curriculum materials have been written. Reports to date indicate that students have been highly receptive to the computer project, and have produced a wide variety of computer programs. These students have also been directly motivated to learn more mathematics, even away from the computer terminals.

The project will continue to investigate three areas: computers as teaching aids, teaching computer techniques to teachers, and the lowering of computer costs. Publication of the project's findings is slated for the future.[11]

Common Threads

Because of diverse goals and varying emphases, it is yet too early to discern an emerging mathematics curriculum. However, among the general characteristics of this fermenting curriculum are the following:

1. Many topics, found to be obsolete, are bing deleted.
2. Many new topics are being introduced.
3. An attempt is being made to teach more mathematics in less time.
4. The concern of many efforts is the utmost development of scientific potential of the superior student.
5. Some efforts are aimed at increasing the precision of mathematical language leading to its clarification and simplification.
6. The student is being provided with an opportunity to participate more vitally while learning mathematics—he is becoming more of a doer than a spectator.
7. The student is expected to develop his ingenuity by *discovering* mathematical relations on his own rather than being told what they are by the teacher; carefully designed sequences of questions lead students to make such discoveries.
8. There is more emphasis on the study and recognition of structural characteristics of mathematics; individual concepts and skills are viewed as parts of larger and more significant mathematical structures.
9. Direct involvement of mathematicians, mathematics educators, psychologists, researchers, teachers, supervisors, and administrators has become an accepted procedure in planning curricular reforms.

10. Financial backing of governmental agencies, private foundations, local school systems, and local organizations has become a form of support for curricular reform efforts.

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Sloppy reasoning and wishful thinking have led some schools to conclude that there is no longer need for and no place for local curriculum-making in mathematics. On the contrary, it is more necessary and more possible than it ever was previously.

Curriculum Development at the Local Level

DON K. RICHARDS

THE development of curriculum and accompanying instructional materials is a continuous process in the field of school mathematics. However, tremendous advances in such areas as mathematical research, the use of mathematics in technology, computers, and automation have greatly increased the speed of curriculum development in recent years. Because of the vast upheaval in parts of the mathematics curriculum during the past decade, many of the fairly well established steps involved in developing a new curriculum or in revising an existing one have been subjected to intense scrutiny by nationally recognized mathematics specialists, and far-reaching recommendations for change in content and methods have resulted.

Curriculum development in the past was generally a joint teacher-student classroom activity conducted during the course of a school year, or it consisted of resource units made available to teachers as needed. More recently, however, entire curricula have been developed on a national scale, by the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, and other similar large development pro-

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grams. At the same time, local school districts and individual schools still undertake some of their own curriculum development activities.

Considerable attention has been given in recent literature to large-scale curriculum development such as that by SMSG, but relatively little has been written about recent smaller-scale curriculum development projects by local units. Once again, local projects are increasing in number. What kinds of choices can and should local units—teacher, department, school, and school district—make regarding the curriculum to be used?

This article will discuss some of the types of decisions that are most appropriately made at the local level in developing sound mathematics curriculum for students and will pose questions for consideration by those engaged in such activities.

Past Pattern of Curriculum Development

What has been the principal pattern of local district curriculum development in the past? Generally, the major goal of such a project has been to provide the teacher with a rather complete package of instruction materials including an orientation to their development and use. Thus, it was possible to say that two major work activities generally constituted the production of such a curriculum. One major activity was concerned with the development and preparation of instructional materials and the other with the process of disseminating information about them.

The preparation of instructional materials frequently consisted of work centered around determining instructional goals and evaluating them from several points of view, plus establishing the structure to be used in organizing the knowledge to be covered from the subject-matter field of mathematics. Distinction was made between the long-range objectives one hoped to achieve and certain short steps designed to move one toward those objectives. Once objectives and structure had been determined, initial material development was begun. Identifiable tasks included the preparation of teacher and student materials, audio-visual materials, evaluation techniques, and additional reference material. After initial materials had been developed, they were subjected to a field test or tryout in a selected sample of schools. Upon completion of these operations, final instructional materials were prepared.

The dissemination process began early in a project by outlining the necessary procedures and by selecting schools even before all the material was ready. When the final materials were available, teachers and administrators not involved in their development were oriented to their nature and use. Lay personnel such as PTA groups were also oriented once the school staff understood the material.

Those fairly well established steps are still being applied in many areas, but local authorities are not as sure of themselves as they used to be. Part of this hesitancy results from the large-scale attention devoted to curriculum development at the research level by many authorities in mathematics. As has been mentioned, there has been an unprecedented participation in school mathematics curriculum planning by university scholars and scientists, men distinguished for their work at the frontier of knowledge. They have been preparing courses of study for the schools reflecting recent advances in mathematics and including bold ideas about the nature of school experience. In consequence, some local directors of curriculum apparently have decided that the period of curriculum-making at the school or district level was over and that the challenge now was to select the best program available that had been developed by the "experts."

Selection of a Program

With each of the large-scale curriculum programs based on a particular approach and promoted by seemingly outstanding mathematicians, the great challenge local leaders saw facing them as they sought to build a sound mathematics curriculum was the selection of a proper program from among the several available.

Mathematics supervisors are frequently asked the question, "Which program is best?" The usual answer is that there is no one "best" program; but this answer is usually unsatisfactory to many questioners, as noted by their facial expression. Supervisors usually try to explain their answer by pointing out that some curriculum programs have involved large numbers of writing participants and have been tested extensively on a national scale while others intentionally have been limited to regional participation and limited authorship. Each program has its own strengths and some weaknesses.

The "best" program for any individual school or district must necessarily be that combination of elements which best satisfies the objectives the local unit sets for itself. When accompanied by appropriate teacher training, this "best program" may be the adoption of a total program or textbook series; it may be the adoption of a single course from one program and another course from another program, the incorporation of elements from one program into another; or the "best" may consist of a selection of useful topics from several of the programs.

SELECTION CRITERIA

Each school or local district has needed some criteria to make a valid evaluation of any particular program it is considering in view of its own needs. As a local unit established such criteria, it has considered such questions as the following posed by a Committee of the National Council of Teachers of Mathematics:

1. How much emphasis should be placed on the social applications of mathematics?
2. At a particular level, what topics can be most effectively developed and which are most appropriate?
3. What emphasis should be placed on the study of mathematical structures?
4. How rapidly should the student be led from the use of the general unsophisticated language of mathematics to the very precise and sophisticated use of it?
5. What is the relative merit of presenting a sequence of activities from which a student may independently come to recognize the desired knowledge as opposed to presenting the knowledge and helping students rationalize it?
6. What relationship should exist in the mathematics programs between the function of developing concepts and that of developing skill in the manipulation of symbols?
7. At what level should proof be introduced and with what degree of rigor?
8. What provisions can be made for evaluating the changes taking place?

The Future Pattern of Curriculum Development

DEVELOPMENT OF A PROGRAM

Recently, many curriculum writers have recommended that individual schools should have greater control over their own curriculum than is presently the case. In looking toward the near future, it appears that local units will achieve greater independence and flexibility in curricular matters. The primary activity of a school or district will no longer be the selection of a program; rather, it will be the development of a program to provide for individual differences among both schools and neighborhoods. Once again, but to an even greater extent, curriculum development at the local level will be emphasized.

Development Criteria

There are a number of different theories and practices popular in curriculum development today. Most mathematics educators agree on the general goals to be sought but there is considerable difference of opinion on exactly how these goals are to be achieved. It is not possible to consider these differences of opinion in great detail in this article, but answers to the following questions have a direct bearing on the type of curriculum developed by a local district or school.

1. What is the mathematics curriculum?

There are at least two broad definitions of the mathematics curriculum. It may be viewed as a collection of "courses" or as written "courses of study," or it may be considered to be all of the mathematical experiences the student has under the influence of the school. A local unit will need to decide whether the mathematics curriculum it proposes to develop is to be viewed in terms of students' experiences or as a collection of topics.

2. What mathematics should be taught?

Much disagreement centers around the content of the mathematics curriculum. Should it be made up of specific fundamental learnings, broad commonly recurring questions, or material necessary for the day-to-day living of students?

Another facet of this question is the order in which learning experiences should be introduced. Many educators believe that the sequence should be determined by the logic or structure of mathematics. Others claim that students do not learn that way. Some maintain that the sequence and organization are to be found not in mathematics but in the learner, and that they should be determined by the needs of the students.

There is also a difference in point of view as to the degree to which the mathematics curriculum should consist of first-hand, concrete illustrations and experiences.

Local developers of curriculum must therefore focus their attention on the content, order, and abstractness of the mathematics program.

3. Should the demands of society or needs of the individual determine the mathematics curriculum?

One point of view holds that since the schools are established by a society to serve its needs, the content of the mathematics program should deal primarily with the needs of that society. The second maintains that since the business of education is to educate the individual, the mathematics curriculum should be built around the needs of the individual. The issue centers around the relative emphasis a local unit should accord the needs of the individual and society. The content of the curriculum will not be an "either-or" stand, but will deal with the individual-in-society. There may not even be a real conflict when viewed by some schools or districts.

Another aspect of this issue is the question of whether the mathematics curriculum should be determined by adult needs (preparation for the future), or by the student's immediate needs. Some local curriculum committees claim that there are enough concerns common to both children and adults to provide a series of learning experiences that will at the same time have immediate significance in the lives of students and prepare them to adapt to changes in the adult world.

4. How much should the mathematics curriculum vary from school to school?

Closely tied to question number three is the problem of uniformity versus diversity. Is it desirable, for example, for all schools in a system to have the same scope, content, and sequence in their mathematics programs? Or, since each community is in many respects culturally unique and since students exhibit wide individual variations, should each school's mathematics program be unique in marked degree? Some claim that there are unifying themes or strands in mathematics which should influence the learning program in all schools. A defensible position, therefore, would seem to be one which calls for a broad curriculum framework consistent with universal needs and conditions, within which a local school or district can vary its program sufficiently to adapt it to the unique requirements of the students within its community.

5. When and by whom should the mathematics curriculum be planned?

This question centers around the problem of the responsibility for determining what the mathematics program shall be. Some believe it is a specialized job to be done entirely and in detail by "experts" in curriculum planning well in advance of the learning situation. Some others believe that the curriculum should be planned by teachers and students on the spot, without any pre-planning.

Local curriculum planners again must determine the optimum position between these extremes. Perhaps a desirable situation would be to have the mathematics curriculum planned in broad outlines using wide teacher and community participation. Then details could be determined through teacher-student planning.

6. How should the mathematics curriculum provide for individual differences?

It is commonly recognized that human beings vary widely in ability. This has encouraged mathematics educators to attempt to devise means of adjusting their program to these differences in abilities. Here again there are different beliefs as to how this should be done. Some promote the concept of a mathematics curriculum established in its minimum fundamentals, with the main problem being that of helping students of varying learning rates acquire these fundamentals.

Another viewpoint is that students should be grouped according to their ability in mathematics, on the assumption that this will reduce the ability range in each group and simplify the job of individualization. A third point of view is that a question-centered, experience-type curriculum permits all members of the group to work together on a common topic in which they are interested, with each class member working at his own level of academic development and yet contributing to the group enterprise.

While the first two viewpoints are the most prevalent in practice, the third appears to be growing in acceptance because individualization must deal with the quality as well as the rate of learning.

7. WHAT IS THE RELATIVE IMPORTANCE OF CONTENT AND METHOD?

A sound approach here by a local school unit will probably recognize that learning is the interaction of the learner with a situation which includes content, materials to help understand that content, and a teacher to help guide the learning through skillful questioning. In this situation, content, instead of being important per se or in relation to its usefulness in the remote future, has value now because it contributes to the present life and understanding of the learners. It helps them solve important problems and gain confidence in their own thinking.

The foregoing are examples of questions being debated by mathematics educators. They include some of the major factors determining curriculum development and organization. The position concerning them taken by a local district or school will determine to a large extent the type of mathematics curriculum developed and put into operation in the future. And, it is to be emphasized, these are questions on which local schools or school systems can and should take positions.

Individual Teacher Choices

Coupled with the trend toward local curriculum development will be the expansion of the teacher's role in making curriculum choices. Increasingly, as the possession of knowledge for its own sake is seen to be less important than the skills for gaining and dealing with knowledge, teachers will gain greater independence in curriculum planning. Teachers who choose to use mathematics as a ringboard for encouraging students toward critical thinking, problem-solving, and research will have considerable latitude in selecting the topics they will use from among many important ones.

How does a teacher select experiences for his students? Teachers do make such crucial decisions daily, using the texts or the new materials they may be acquainted with as sources of topics. It is necessary that they have some criteria in mind to aid them in making these decisions. A positive response to the following four questions will generally assist the teacher of mathematics to make a better selection of mathematical topics and experiences:

1. Is the topic a fundamental concept rather than just a technique?
2. Will the students play an active role?
3. Is there opportunity for students to make discoveries?
4. Is the idea appropriate for students' needs at their stage of development?

While each of the many new mathematics curricula formulated during the past decade has unique features, most have some elements in common. Therefore, the teacher might first ask himself, "Is the idea I'm considering a fundamental concept or a technique? Is it significant to the field of mathematics and common to most of the suggested curricula?" Teachers want to offer their students the best and most genuine mathematical experiences they can devise, and they want these experiences to deal with basic concepts of mathematics.

The second question teachers might ask themselves in considering the development of an idea or the selection of a experience is, "Will the students play an active role?" Educational psychology stresses the fact that the best motivational device for effective learning is self-actualization. Teachers can provide for a student's natural desire for "activity satisfaction" through the constructive and creative use of his talents and can help him find opportunities in his learning of mathematatcs for constructive self-expression in social situations.

A classroom was recently observed where the students had the chance to be very active. A rubber band had been suspended from the top of the chalk board and different students took turns hanging gram weights from it and observing the corresponding reading on a meter stick taped to the board. One student would make a reading, another would attempt to verify it, a third would put the resulting measurement in tabular form on the board, while still others prepared a graph of the results. Everyone had a chance to participate actively several times in the project, and such mathematical concepts as approximation in measurement, function, and graphing took on real meaning for the group. Perhaps projects like that one cannot be going on all the time but certainly more such activity is needed.

As to the third criterion for curricula selection—"Is there opportunity for students to make discoveries and generalizations?"—Aldous Huxley, in one of his books, quotes a poem which demonstrates a philosophy not yet dead in connection with mathematics instruction:

> Ram it in, ram it in!
> Children's heads are hollow.
> Ram it in, ram it in!
> Still there's more to follow.

In the most rapidly changing society in history, teachers can no longer foresee what a student will need to know. All the teacher knows is that education will be a lifelong activity, so the most important things he can teach are methods of acquiring new knowledge. Discovery is one of the best such methods and students can learn it most easily by applying it. Teachers may help through the wise use of leading questions, by provision of appropriate materials, by encouraging and rewarding problem-solving, and by being particularly careful to offer guidance only as needed. Students find discovery-learning fun and their self-esteem is enhanced by it. A memorized formal definition tells a student what something is only if he already knew it.

Finally, teachers should consider whether the topic is appropriate to student needs.

There is the old story of the man who entered his son in school by saying "I don't care what you learn him as long as you learn him Greek. I want him to learn Greek for three reasons: first, because Greek is hard; second, because he don't wanta learn Greek; and third because Greek'll never be a damn' bit o' use to him!" Some teachers appear to use somewhat similar reasoning when helping students enroll in mathematics courses or when selecting topics or experiences for student consumption.

Other criteria for selecting subject-matter include, for example: (1) Has the mathematical concept stood the test of time? (2) Is the idea useful or can it be applied? (3) Is the topic interesting to the learner? and (4) Will the material contribute to the student's growth and development in understanding a body of knowledge making up a branch of mathematics? The point is, curriculum choices will be made by individual teachers and must be based on valid and carefully articulated criteria.

Summary and Conclusion

Because of the recent turmoil in mathematics education, some local school units may be tempted to wait for more general agreement among mathematics educators. This would be a mistake and many schools and districts realize this. They know that their present teachers are the key to a successful mathematics program. Administrators can encourage their teachers by showing interest in the mathematics curriculum. They can make sure that their teachers have access to new materials, professional journals, workshops and other in-service resources so that they may become adequately trained to recognize different points of view and thus make wise decisions based on content, not just on the superficial use of new language, which characterizes many so-called contemporary texts.

It is important for local leaders to notice the emphasis placed on teachers being adequately prepared. Curriculum is really a " people" problem. It is not possible to change the curriculum without changing teachers. They must be acquainted with various mathematics curricula *before* they can develop their own specific program.

The role of each school or local district seems clear: to examine critically the approaches suggested by various study groups, commercial companies, and other local units, and to make use of those elements that satisfy its own particular requirements. Local units cannot abdicate their responsibility for curriculum development by waiting for others to do the job for them. Commercial textbooks are not the answer. They, along with state curriculum guides, can only assist.

Generally, the state will provide a broad curriculum framework within which each local district or school can work. Frequently this is the state's legal responsibility. This framework will serve to delimit the area and suggest those concepts and methods all students should have had some experience with in a particular course. Local units can then design their own curriculum within this framework, adapting it to any unique requirements of the community. The final details can best be determined in the classroom through teacher experience.

The four papers that follow show specifically how four school systems have gone about making their local decisions and what the outcomes of these decisions have thus far been.

The revolution in mathematics has significantly influenced both outlook and practice in independent as well as public schools, as this description of recent events in the academic life of one school demonstrates.

Mathematics at St. Mark's School of Texas

W. K. McNabb

St. Mark's School of Texas is an independent day school with selective admissions. It has a maximum enrollment of 688 boys in grades 1 to 12. The school is separated into three administrative divisions. The Lower School includes grades 1 to 4, with a maximum of 128 boys and a separate faculty. The Middle School contains grades 5 to 8 with a maximum of 240 boys, and the Upper School has grades 9 to 12 with a maximum of 320 boys.

Mathematics in grade 1 is taught within self-contained class rooms, but mathematics in grades 2 to 4 is taught by mathematics specialists. One of these mathematics teachers coordinates the mathematics program of the Lower School under the direction of the head of the mathematics department. The nine members of the mathematics department teach all mathematics courses for grades 5 to 12 on a two-year rotation of course assignments. New men in the department usually begin at grades 11 and 12 and then work their way down through the curriculum. After reaching grade 5, they begin the cycle again at grade 12.

The mathematics program of grades 1 to 6 follows the Singer Series *Sets and Numbers* by Suppes, supplemented by some locally

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prepared materials and by introductory work with a slide rule in grades 5 and 6. Grades 7 to 11 follow an integrated spiral sequence of elementary mathematics materials. Currently these consist of a locally prepared text for grade 7, followed by the grades 9 and 10 texts of the Copp-Clark Canadian series in grades 8 and 9, the SMSG text *Geometry With Coordinates* in grade 10, and SMSG *Intermediate Mathematics* in grade 11. Present plans are to phase out the SMSG texts during the next two years, replacing them with the grade 11 and 12 texts of the Copp-Clark series.

Honors Work Offered

Two levels of courses are offered. Each year one or two of the five sections in each course for grades 5 to 12, depending upon available students, become honors sections. These classes study the same materials as do other students and take the same examinations, but the depth and tone of treatment is different and a few extra topics are introduced, particularly in grade 11.

Mathematics is not required of St. Mark's students in grade 12. Students of grade 11 honors sections are eligible for the APP mathematics course of the CEEB in grade 12. These students are also eligible for two one-semester courses, one in Matrix Algebra and the other in the SMSG Analytic Geometry. The grade 11 nonhonors students or others not selected for the grade 12 honors courses take in the twelfth grade an elementary functions course using the SMSG texts, *Elementary Functions* and *Intermediate Mathematics* (chapters 11 to 15). For 1968-69 it is planned to shift these students to the new AB-level course of the APP of the CEEB.

For those students who do not plan to take any college mathematics, an introductory calculus course of one semester is offered. A one-semester probability and statistical inference course is also available, and it is planned to add one-semester courses in numerical analysis and number theory within several years. Also in 1968-69 we plan to change our present APP calculus course to the new BC-level course of the APP.

For students particularly interested in mathematics, Problems Seminars are offered for both Middle and Upper School students. These seminars consider more difficult problems than those that can be used in course work. The Upper School group makes considerable use of the Olympiad problems of Eastern Europe and others of related difficulty. Materials for the Middle School group are much more difficult to obtain.

A wide range of mathematics content is available in the central library of the school with over 150 volumes excluding mathematics textbooks kept for reference in one of the mathematics office areas. The school also has a chapter of Mu Alpha Theta, the national honorary society for high school and junior college mathematics students, and each year approximately 150 students participate in the Annual High School Mathematics Contest sponsored by MAA, the Society of Actuaries, Mu Alpha Theta, and the NCTM. The school also sends entries each year to the Hockaday and Andrews Mathematics Contests, and it sponsors a Junior High School Mathematics Contest for students in the Dallas metropolitan area. Midyear and end-of-year papers are required of students in grade 12 honors courses.

Entry Test in Math

As a part of the general admission process for students applying for entrance to grades 5 to 12, a mathematics test is given. The result of this test is reviewed by a committee of the department to determine the student's ability and background. A recommendation concerning possible admission is then made to the school admissions committee. If a student is accepted for admission, he is then placed at an appropriate level of the mathematics sequence by the mathematics department regardless of his general grade placement. A summer school program is available and recommended for new students as a means for making early adjustment to the mathematics program of the school.

Each spring a standardized achievement test is given to all students in grades 3 to 11 in order to relate local students achievement to that of other schools. The results of these tests are also used to develop local norms which are used with the mathematics admissions test. Also, each spring a detailed questionnaire is sent to all of our previous year's graduates in order to check their secondary mathematics training against their college mathematics experiences.

The mathematics department is housed in two wings of the Mathematics-Science Quadrangle. Six members of the department have individual offices around a central common room in one wing, while four others share a large office area in the other wing. Electric typewriters with interchangeable keys for special symbols are available in each office area, and a supply room contains duplicating equipment: mimeograph, spirit duplicator, and photo copy. Other equipment includes a six-stage collator, electric stapler, Thermofax copier, two portable overhead projectors, and file cabinets containing specially prepared overhead projector transparencies. The hall outside one office area contains two large bulletin boards and 12 framed pictures of famous mathematicians, together with a brief bibliography.

Logistics and Organization

Eight classrooms and a seminar room are contained in the mathematics wings and all mathematics classes for grades 5 to 12 meet in these rooms. Each room is fully carpeted, has four walls of chalkboard, and has separate desks and chairs for flexible seating arrangements. At one end of each classroom is a three-deep set of sliding chalkboard panels including both rectangular and polar grids. A projection screen, large demonstration slide rule, and closed circuit television system is also available in each class room. The seminar room contains two large conference tables. The department also has nine Monroe Educator desk calculators which are kept at the rear of one classroom for student use during any free time when the machines are not in class use.

In the common room of the mathematics department a statistical-type rotary calculator and a printing calculator are available for faculty and student use. Present plans are to extend the use of calculators, which now begins in the Lower School, by the addition of a four-terminal, high-speed electric calculator and a small computer in the very near future. Work with computers is not a part of the mathematics program, although several short courses in programing have been given on Saturdays by staff personnel of local commercial firms.

The teaching load of a department member is four sections of not more than 16 students, each meeting daily five times a week. An effort is made to split the courses of each teacher into two closely related grade areas. In addition to this, a teacher is expected to carry some special department duty as well as an assignment involving a general school activity, such as sports or special interest groups. Special department activities involve handling problems seminars, contests, library, bulletin boards, and audiovisual materials.

Teaching assignments are made each year so that two or three teachers are involved in each course having more than one section. One of these teachers is designated as course coordinator and is responsible for the preparation of lists of daily assignments, which are printed and distributed to the students, and for preparation of the unit examinations. A full-period examination is given at least once within each two-week period. This examination is cooperatively prepared by all those teaching the course. A common scoring process and grading scale is agreed upon for each examination and is used for all sections. Copies of the assignment sheets and examination. together with grading methods, grade scales, and error summaries are filed for future reference.

Every second week that does not coincide with a school grade report time, a departmental checklist report is turned in to the department head for all students having difficulty or working below their capacity. Copies of these reports are kept in the mathematics office, and the reports are given to the heads of the Middle and Upper School for information. They in turn pass these on to the students' advisers for information and action.

In-service Training

So that teachers continue to develop professionally, a departmental seminar is held periodically with faculty from both public and independent schools in the area invited. These meetings usually consist of an informal presentation of some topic in mathematics which is of special interest to one of the group, or a discussion of new materials, new equipment, or new methods. Participation is voluntary and open to other department members also. In past years formal study of a topic or text was tried, but it was not as successful as the current practice. An effort is made to send each department member to at least one regional or annual meeting of the NCTM during the year, particularly outside of the local region. This has been a great help in keeping department members aware and informed about new materials, curricula, and methods.

This program in mathematics has gradually evolved, and is continuing to change each year even more rapidly, from a rather drastic change made in 1960-61. At that time, the full impact of the new programs in mathematics was being felt and a decision was made, chiefly on the basis of the availability of a person familiar with the SMSG program, to create a department of mathematics specialists and to move immediately into a "new" mathematics program. Up to then, a traditional sequence of algebra, geometry, and trigonometry had been taught by science teachers, administrators, coaches, and only two teachers prepared as mathematics teachers.

In the fall of 1960-61, grades 7 to 9 moved immediately to SMSG texts, with grades 10 and 11 both doing a traditional geometry course in order to change the following year from the algebra-algebra-geometry sequence then used in grades 9 to 11. The APP mathematics course of the CEEB was begun this first year, with the compromise that those students would also take an elementary functions course. Trigonometry and solid geometry were dropped from the program. The excitement of new materials and both the enthusiasm and hard work of the faculty managed to carry off this drastic change. During the past six years, changes in the Lower School mathematics program began affecting later courses severely. The SMSG program for grades 4 to 6 was adopted the second year (1961-62) and this change soon put pressure for change with the grade 7 material. Similarly, the introduction of the Singer materials in grades 1 to 3 soon forced out the grades 4 to 6 SMSG texts. By 1965-66, the SMSG program in grades 7 to 9 began to be phased out due to the strong preparation of students in the lower grades.

In Review

In considering the changes since 1960-61 in the St. Mark's School of Texas mathematics program, it is very difficult to find any clear-cut basis for the many decisions that were made for change. For the most part changes seem to hinge more strongly on the personalities involved than on logical or scientific bases. However, a few general points seem evident in relation to these changes and those still in progress:

- Current mathematics programs demand highly trained specialists with strong mathematics background at all levels.
- Students can develop a thorough understanding of much more sophisticated mathematics content if it is presented in a spiral manner with emphasis on structure and relation to previous material.
- Continual adjustment of the curriculum is necessary as new materials and methods appear.

- The enthusiasm and interest of students in mathematics is not dependent upon the use of social or scientific applications.

In 1967-68, a five-year plan for modification of the St. Mark's mathematics program was projected; now, one year later, much of it has already been changed considerably. It appears that mathematics education at the elementary and secondary level has now reached a continuing state of change, so that curriculum and methods studies have become an important part of the mathematics department operation.

The chairman of the mathematics department in a school that has been a leader in the curriculum revolution says. "If we are not careful, we may get into a rut and teach the new mathematics in as sterile a way as we taught the traditional mathematics."

Mathematics in Newton

W. EUGENE FERGUSON

THE complete story of the mathematics curriculum in the Newton, Massachusetts, schools cannot be told in a few pages. The curriculum is a dynamic one that is unfolding year by year, and by the time a curriculum guide can be put down on paper it is obsolete in some respects. Experimentation, the development of new content and new courses, and the development of new techniques and new approaches to teaching mathematics never cease.

Any complete story must consider a kindergarten to grade 12 curriculum, so in this short story I shall concentrate on the development of the program in grades 10, 11, and 12, but indicate the supporting roles of the 25 or so elementary schools and the five junior high schools. I cannot overemphasize the point that our senior high school program could not possibly have developed to its present level without the corresponding development of a modern program in the elementary schools and the junior high schools.

I came to Newton as head of the mathematics department in 1955, so I am going to use that as my base point. I shall develop this story in four parts:

- Description of 1955 program.
- Description of 1968 program.

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- A short story of the 13-year development from the 1955 program to the 1968.
- A crystal-ball look at the developing program during the next five to 10 years.

The 1955 Program

In 1955 the elementary schools used a curriculum guide that was ahead of its time. The junior high had essentially the standard curriculum that was excellent for 1955. The high school had some 2,600 students, about 1,200 of whom were studying mathematics, and 12 mathematics teachers. There were essentially four tracks.

Track 1. About 25 of the academically talented students in mathematics were chosen to complete geometry and second-year algebra in one course in the tenth grade. Courses at Newton meet four times a week in approximately 50-minute periods, so this course was "integrated" by studying algebra two days a week and geometry two days a week. The junior year was a precalculus course in topics from college algebra, trigonometry, elementary functions, and some analytic geometry. The senior year was the CEEB advanced placement course in analytic geometry and calculus. Obviously, a year had to be gained in order to teach calculus and analytic geometry in grade 12. It was also clear that a speeded-up program such as this could not be handled by any but the few academically talented in mathematics.

Track 2. This was the standard college-preparatory program with one excellent innovation. The sophomore and junior years were considered a two-year sequence, the idea being to teach second-year algebra and geometry over the two-year period. This was done quite successfully by teaching, for example, algebra on Monday and Wednesday and geometry on Tuesday and Thursday. This meant that algebra and geometry were carried along together so that each one could support the other as much as was possible using the books and materials available then. This avoided the problem of students' forgetting ninth-grade elementary algebra while studying geometry in grade 10 and avoided the usual massive review of elementary algebra in grade 11 before second-year algebra topics could be started. Teachers felt that students had more algebra and geometry at their fingertips at the end of grade 11 and were better prepared for the senior course as well as the various College Board exams. The senior course was not quite standard for 1955, because solid geometry was disposed of in six weeks and the rest of the senior year was spent on trigonometry, college algebra, and analytic geometry.

Track 3. This was a plane geometry course for those who could not manage Track 2 and for whom it was the end of college preparatory mathematics. There was an elementary algebra course for those who failed in grade 9 or took general mathematics in grade 9.

Track 4. This was a noncollege-bound track that had a basic or general mathematics course with social applications and an applied mathematics course that was taken almost entirely by boys who expected to do apprentice work in many areas after graduation.

So we see that in 1955 students in Tracks 1 and 2 had an opportunity to take a mathematics course in each year in high school. In Track 3, plane geometry was the most advanced course available. In Track 4, the students could get a year of general math or a year of applied math.

The Program Today

Now for a description of the 1968 program. In Newton High School, 20 teachers offer 26 courses in mathematics and two seminars to about 2,000 students. At Newton South High School, 16 teachers offer 18 courses to about 1,600 students.

Each elementary school develops its own program; they have used and are using materials developed by the School Mathematics Study Group, Greater Cleveland Math Program, Madison Project, Illinois Arithmetic Project, and others. They are also using commercial texts based on the many experimental programs developed during the last 10 years.

The junior high has developed a more flexible scheduling procedure so we have students in the mathematically talented group studying algebra in grade 8 and geometry in grade 9. In grade 9 most schools teach the integrated grade 10 program, half geometry and half intermediate algebra. The junior highs are using many of the "modern" hard cover texts.

The programs in the two senior high schools are essentially the same except for the trying out of various courses at Newton High School to meet a wider range of student ability and interest.

There are essentially five sequences or tracks, together with a number of courses that don't fit exactly into any sequence. Track 1 consists of two honors sequences, one leading to the CEEB Advanced Placement BC exam and the other to the AB exam. Until 1969, these students will all take the standard present AP exam.

It will take three or four years to get these BC and AB programs into complete operation, but we expect to have a few students ready for the 1969 BC exam even if we have to do a few topics as independent study.

Pre-calculus Program

The present Track 2 program leads to calculus in college during the freshman year. Grades 10 and 11 are an integrated sequence that covers in grade 10 the first half of geometry—both plane, solid, and some coordinate geometry—and the first half of intermediate mathematics, with the last half of each course in grade 11. Intermediate mathematics includes second-year algebra and trigonometry. The two courses are integrated in such a way that the algebra needed in the geometry is studied just before it is used in the geometry. This isn't the ideal way to integrate these two courses, but until textbooks are written in this manner it is the best we can do. Classes meet four periods a week and it is possible for teachers to teach a unit of algebra and then a unit of geometry, or to alternate a day of algebra with a day of geometry as they wish. This is easier to do today with the "modern" textbooks than in 1955 because much algebra has been put in the geometry course and more geometry is in the algebra course.

With geometry and intermediate mathematics taught over a two-year period and the resulting saving of review time, it is hoped that we can cover all the topics in the present-day geometry and algebra-with-trigonometry books. In other words, we feel that this integration makes for more efficient teaching. The senior course in this track can now be devoted to pre-calculus topics including elementary functions, vectors, analytic geometry, limits, series, and possibly an introduction to calculus. This program prepares the student to take a calculus course as a freshman in college. Many colleges and universities now have calculus as their first mathematics course. This means that analytic geometry must now be a high school subject for many college-capable students.

The Track 3 program is a college preparatory program with the geometry and intermediate mathematics courses integrated in the sophomore and junior year in the manner of Track 2. This integration has been talked about for a number of years and it starts with the 1968-69 school year. We hope that we will be able to find textbooks that have much the same content as for Track 2 but at a lower reading level and that also do not go quite so deeply into each topic. Of course, some topics will be left to the senior year, which will consist of trigonometry, some analytic geometry, some topics from college algebra, and elementary functions. At the present time these students are prepared to take a freshman pre-calculus course of one or two semesters. With the gains from the integrated geometry and algebra we hope the track will get stronger during the next five years. I should add that it is very difficult to find a good textbook for the senior year of this track

For the Noncollege-bound

Track 4 is the the noncollege-preparatory track. Basic Math 1 and Basic Math 2 in this sequence are essentially pre-algebra courses and include arithmetic, geometry, and algebra. (These are essentially good modern seventh and eighth grade programs.) Consumer math for seniors only—a half-year course offered each semester—contains the social applications and remedial arithmetic needed to solve everyday problems of the adult in our society. The basic philosophy of the mathematics department is that a course must be available to any student each year he is in high school, no matter what his level of ability and achievement in mathematics.

Since the Basic Math courses are slanted toward pre-algebra and possibly entrance to Track 3 for some students, we are starting in 1968-69 Applied Math 1 and Applied Math 2 that will contain as much of the structure of mathematics as the students can take; but these courses will be slanted toward the "practical" mathematics used in many apprentice training programs. Hopefully these courses will be taught cooperatively by a mathematician and a shop man. Plans are to spend possibly two days each week on the "why" and the structure of mathematics and two days on practice using problems of interest to each student;

that is, a laboratory approach with more individualized instruction. This, we trust, will interest more students who now see little use for mathematics as we teach it in the college preparatory sequences.

Courses in the college preparatory Tracks 2 and 3 start in the fall and also in the spring semester, so that at any time we have the first half and the last half of the courses being taught. This gives us great flexibility in moving students from one track to the next as well as giving students an opportunity to start over in a course without waiting until the fall semester. A student no longer needs to sit all year in a mathematics course that he does not understand.

Supporting Courses

I shall now describe the supporting courses and other courses not in the sequential tracks although they are in the branches off each track. Even though there is usually a normal year in which each course is taken, a student may elect any course in any year that he is prepared and qualified to take it. Some students are seniors before they have developed sufficiently mathematically to take elementary algebra, for example.

Elementary algebra is offered to students who are a little slow in mathematics.

Logic will be offered starting in 1968-69 as a half course, meeting twice a week for a year. This course is principally for sophomores and juniors. The flexibility of meeting some courses twice a week for a year allows students to take music, art, or some other two-day-a-week subject that meets at the same time of day, e.g., math on Mondays and Wednesdays, art on Tuesdays and Thursdays.

Computers meets twice a week for a year and a console (teletype machine) is available for time-sharing on a GE-265 computer. Students program their problems in any of three languages—Basic. Fortran, and Algol. A key punch is also available so that students may punch cards and have their program run on some computer at a university or industrial plant.

History of Mathematics, which will be offered for the first time in 1968-69, meets twice a week and will be open to any students in a college preparatory track.

Probability and Statistics is available to seniors in Tracks 1 and 2. It is a first-semester course that meets four days a week.

Matrix Algebra is the companion second-semester course. Some students take these two courses instead of calculus, some take these two courses and also calculus. Some take the Track 2 senior course and one or both of these semester courses simultaneously.

A seminar in algebra is offered one period a week and also a seminar in analysis. These are run like college seminars and are for those few students who desire to do independent study on certain topics and then report to the group.

Semester courses are offered in elementary algebra, geometry, and intermediate algebra for those students who have taken one of those courses once but did not learn it too well and now really want to acquire a command of the subject. Juniors might find one or more of these semester courses helpful in pulling them through the integrated sequences in Tracks 2 and 3. These semester courses give a student, a senior in particular, a chance to learn some mathematics that he would skip if he had to repeat a whole year to get it.

So we see that in 1968 each student has available to him one or more mathematics courses at his level of development in each of his high school years. This is in line with the philosophy of the department that students in this technological age need all the mathematics they are capable of learning.

In 1955 about 50 percent of the students were enrolled in a mathematics course; in 1968 about 80 percent of the students are taking mathematics. There are many reasons for this growth in enrollment; one of these, of course, is the mounting needs of the technological age. But a very big reason, we believe, is the availability of courses geared to the greatly varied levels of ability and achievement of the students. In Newton High School the change from 1,200 to 1,900 taking mathematics was the result almost entirely of the availability of three years of mathematics in Track 3; almost all of the 700 more students were in Track 3, while Tracks 1 and 2 have remained about constant. (Of course, the need for more mathematics must go hand in hand with the availability of courses at the proper level.)

From Then to Now

How did we get from the 1955 program to the 1968 program? In 1957 the UICSM (University of Illinois Committee on School Mathematics) program was introduced in the junior and senior high schools. In 1959 the SMSG (School Mathematics Study Group) program was introduced in the junior and senior high schools. These are the two forces that have helped to mold and develop the secondary school mathematics program. In the elementary schools, some introduced materials from SMSG, GCMP (Greater Cleveland Mathematics Project), Miquon (Pennsylvania) Madison Project, Miss Mason's School, and from publishing companies.

Many of the junior and senior high teachers attended UICSM Institutes at the University of Illinois and NSF Institutes at various universities. For almost ten years I conducted in-service courses on the UICSM, SMSG, and Advanced Placement programs for our junior and senior staff and some teachers from neighboring schools. Much time has been spent by the Newton mathematics staff learning the "new mathematics" and the teaching techniques involved.

The staff had to become informed about the "new math," but so did the parents. So, we spent many sessions with school PTA's, PTSA's, and just interested parent groups describing and illustrating the "New Math." Even today, if drastic changes are made in a mathematics program the problem of educating the staff and parents must not be overlooked.

We have always had an SMSG track and UICSM track at the high school, but today the SMSG and UICSM books have been replaced by commercial texts whose writers were much influenced by SMSG and UICSM. The UICSM track has become the second honors track—the AB sequence, but we do not use much of the UICSM material. We use mostly UICSM- and SMSG-influenced hard-cover commercial textbooks.

The changes were made continuously; big changes all at once were avoided as much as was possible. Changes are still being made—you have noticed that we have new courses scheduled to start in 1968.

To make necessary changes each year we must consider the K-12 program. Particular attention must be paid to the articulation points such as in a 6-3-3 system from 6 to 7 and from 9 to 10. The selection of new textbooks is no small task.

It has been essential to keep communication lines open among all administrators. Principals must understand the needs for a new mathematics program and their supporting role in the development of a program. A coordinator of mathematics can help very much in keeping teachers and administrators talking and planning together. We now have such a coordinator.

The development of the staff and the hiring practices also help make or break a mathematics program. Colleges today are training teachers who will be unhappy unless they are teaching a modern program. This is quite a change from the late 50's when graduates were still unacquainted with the "new math."

In Years to Come

Now for Part 4 of this story—What do we see in the crystal ball for the next five to 10 years? If we are not careful, we may get into a rut and teach the new mathematics in as sterile a way as we taught the traditional mathematics. It is quite clear that we must make efforts to provide even better for individual differences. We must have opportunities for individualized instruction and provide the opportunity for students to proceed at more nearly the speeds that are best suited for them. Different media must be explored.

With increased attention to developing each student mathematically at somewhere near his optimum rate, we are going to find greater spread of achievement at each grade level. Administrative techniques will have to be found that will allow more students in elementary school to do work in mathematics normally done in junior high, more students in junior high will be doing work normally done in senior high, and so on. The junior high will develop scheduling techniques to take care of the wide spread of achievement of students coming into the seventh grade and will build on their advanced knowledge and not mark time while other students catch up.

A more complete integration of mathematics will occur, and SMSG is working on such a sequence now. The K-12 program in the very near future will find calculus as a standard twelfth-grade course at different levels of abstraction. At the same time that college preparatory mathematics is being "beefed up" more attention will be given to the mathematics sequence for those who will go to work with little formal education beyond grade 12.

Much algebra and intuitive geometry will be done in the elementary schools. The junior highs will build on this program, and geometry will end up as about one semester of work at the most in high school.

Some students will continue to have real troubles in taking mathematics and these will get more attention in the future.

Calculus has been the big push for high school, but in the future, probability and statistics, linear algebra, and matrices will be more generally taught.

The structure of mathematics will be stressed, but many students will take advantage of courses that lead more directly into the world of work.

I have no doubt that the next 10 years will see more and greater changes in the mathematics programs, but the crystal ball is fading a bit and I prefer not to try to see more and thus raise the issue of credibility.

"The most significant feature of our story is probably not the current status of our mathematics program. . . . The most significant feature to me is that we have arrived at our present status through a mixture of human relations and mathematics which provides a sound basis for ongoing study and revision."

Case History, Charlotte-Mecklenburg

JOHN F. SMITH

SINCE organizational structure is a factor in any changing educational practice, let me set the stage for this case history by describing the organizational framework within which our activities have been carried on. Until July 1, 1960, the Charlotte City Schools and the Mecklenburg County Schools were separate administrative units, each responsible directly to the State Board of Education. (In North Carolina there is probably more centralized direction of curriculum and materials than in most other states. There is, for example, a state-adopted basal textbook for each course in the state program of studies. There is also a state outline or synopsis for each course.) Effective July 1, 1960, the two systems were consolidated to form the current organizational structure, a single administrative unit called the Charlotte-Mecklenburg Schools. This system uses the 6-3-3 pattern. There are 11 senior high schools, 20 junior high schools, and 77 elementary schools serving 80,000 students.

There has been no significant change in the relationship between the administrative unit and the State Board of Education; but, both before and after consolidation, the state board has

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encouraged experimentation and allowed considerable local latitude in the development of curricular patterns. State control has not, therefore, been a restricting factor. On the contrary, we have received encouragement and assistance in developing in-service programs and assessing materials.

In developing this case history, I shall first describe the mathematics program of 1958-59, which was our last completely "premodern" year and then turn to the program as it appeared in the school year 1963-64. It was then that the various experimental courses and materials in use were drawn into a systematic sequential pattern. After this comes a description of our current program. Having reviewed program changes, it will be appropriate to look at trends in mathematics enrollments as a basis for estimating student reaction to curriculum revision. Following this, I want to describe the various transitional stages in terms of how they developed, and who developed them. Finally, I'd like to invite the reader to join me in a look at the future. This latter idea implies, quite appropriately, that here in Charlotte-Mecklenburg we recognize the pressing need for constant evaluation and that we do not think the job is or ever will be finished.

The Program 10 Years Ago

In 1958-59, the typical college preparatory program in our high schools included Algebra I, Algebra II, geometry, and Algebra III, taken in that order in grades 9 to 12. In addition, most of our high schools offered semester courses in trigonometry and solid geometry. Students who planned mathematics, science, or engineering majors in college were advised to take these latter two during their senior year. Those who did so had to take two mathematics courses throughout their senior year. Every student was required to have one unit of mathematics in senior high (grades 10 to 12); those who were not taking college preparatory work usually obtained this unit via general mathematics, in which the emphasis was on remedial arithmetic and simple applications.

Changes in 1963

In 1963-64, the typical college preparatory program for grades 9 to 12 included Algebra I, geometry (integrated plane and solid), Algebra II, and algebra-trigonometry (integrated). The basic content was essentially that which made up the program in

1958-59. There were, however, three differences worth noting: (1) geometry was placed between first- and second-year algebra, (2) there was an integration of plane and solid geometry into a single course and an integrated approach to algebra and trigonometry, and (3) the student did not need to take two courses simultaneously in order to get the full program. At that time some students were taking Algebra I in grade 8 and thus were in a position to complete the whole sequence during their junior year. These students were offered a course in functions at the twelfth-grade level.

The program also permitted gifted students to omit the Algebra II course, to take the functions course in grade 11, and thus to be eligible for advanced placement in grade 12. This decision was left up to the individual school. The provision was never implemented to any significant degree because most teachers felt that the omission of second-year algebra left even the gifted student with a dangerous gap in his background as he approached the more sophisticated algebra involved in the algebra-trigonometry course.

The Program Today

The program for the typical college preparatory student today is essentially the same as the typical program in 1963-64. The most significant difference lies in the suggested variations. Advanced classes (not gifted, but well above average) follow geometry with a course which offers an enriched approach to intermediate algebra and a substantial introduction to trigonometry. These students will omit the algebra-trigonometry course and proceed to a course in elementary analysis. The latter is an extension of the functions course and is designed to complete the student's preparation for calculus. The last two courses in this sequence include all the essential features of the algebra-trigonometry course. In other words, for these students we are attempting to present the essentials of Algebra II, algebra-trigonometry, and elementary analysis in a two-year sequence instead of a three-year sequence, thus eliminating overlap and repetition.

The biggest contrast between the current program and the preceding ones is perhaps the change in general mathematics. The requirement of one unit for graduation remains the same. However, the emphasis is shifting toward an approach which draws attention to mathematical structure, and students have responded to this quite well. One of our schools now offers three years of general mathematics, and a few of the others offer two years and are preparing to offer a third year whenever the demand is sufficient.

Trends in Course Enrollment

The percentage of our senior high students enrolled in mathematics courses is on the increase. While we have not established a causal relationship between curriculum revision and enrollment, the figures are worth noting. The year 1962-63 was the last year before an effort was made to systematize the use of modernized materials in our program. It also happens to be the earliest year for which statistical data relating to enrollments is available. During that year, 63 percent of our senior high students were enrolled in mathematics courses. In 1966-67, 69 percent of our senior high students took some course in mathematics. During the current year, 1967-68, 71.7 percent of the students in grades 10 to 12 are taking mathematics. We know that some of this increase is the result of an expanding general mathematics program. For example, over 300 senior high students are enrolled this year in general mathematics courses beyond the graduation requirement.

There has also been an increase in the percentage of seniors taking a course beyond the algebra-trigonometry level (functions or analysis). In 1962-63, 1.3 percent of our seniors took courses at this level. In 1966-67, this figure went up to 3.3 percent, and in 1967-68 it has risen further to 4.3 percent. There has been, then, increased participation in general mathematics and in advanced level college preparatory work.

Transition: People and Process

Now I turn to the process of transition, where the story begins in the spring of 1959. The assistant superintendent for instruction in the Charlotte City Schools had noted an increasing number of references to the new mathematics in the professional literature. Unlike some administrators, he avoided the error of rushing in and issuing a decree that we should henceforth have "modern mathematics." Instead, he called together a committee of mathematics teachers, including several with NSF summer institute experience. He then issued not a decree but a professional challenge. This challenge consisted essentially of the observation that he and the teachers were obligated to study the question and decide what implications, if any, the movement in mathematics had for the school system.

It is very likely that the nature of this beginning step helped to avoid much of the frustration and insecurity that seem to beset teachers whenever curriculum revision comes from above rather than from below. The use of positive leadership and professional challenge established an atmosphere in which mathematics teachers could play a significant role in determining curriculum patterns and perform this role without feeling pressure.

The committee of teachers reviewed the available information during the latter part of the 1958-59 school year. Major attention was focused on the recommendation of the Commission on Mathematics. The committee members felt that the least dramatic change implied by the Commission's recommendations was in the area of geometry. They further decided that it would be feasible to supplement our existing geometry program in such way as to take a significant step toward the program suggested by the Commission. Finally, the committee outlined this proposed supplementary work and offered to develop the necessary materials for distribution to those geometry teachers who might like to try them.

After reviewing the committee's report, the assistant superintendent conferred with a university professor who was known to have considerable interest in the new mathematics movement. He suggested that if the local teachers could go that far on their own, the school system would be a logical one to provide a center for the field testing of the first version of SMSG geometry. He also passed along this suggestion to the SMSG people. Both SMSG and the local system agreed, and in the school year 1959-60, the first modernized text was used. Five senior high schools, seven teachers, and 200 students were involved. Although legal consolidation was two years away, the experiment involved schools from both the city and county systems.

While the experimental course was being taught in the classroom, the committee of teachers (now called the Mathematics Steering Committee) organized some in-service workshops for other teachers. As a result of this, participation increased sharply during 1960-61, with the first revision of the SMSG text in geometry providing the material. Meanwhile, the in-service workshops continued, and the other part of the cycle, increased teacher interest and participation, followed for the 1961-62 school year. During that year some 18 teachers and 1,000 students used SMSG or other modern materials. Twelve schools were now included, because the movement had spread to include some junior high teachers, and some senior high courses beyond geometry.

To spare the reader any further statistical data it is enough simply to point out that the cycle continued. During a given year, while some teachers were teaching experimental materials, increasing numbers of other teachers were undergoing voluntary in-service work. On each occasion substantial numbers of the latter asked to be included in the experimental program for the following school year.

Articulation of Courses

Quite possibly by now you have a feeling that our mathematics movement "grew like Topsy." As a matter of fact, up to the school year 1962-63 there was no consistent pattern, except for the heavy reliance on grass-roots (teacher) readiness. During the latter part of the 1961-62 school year, the Steering Committee became concerned about the possibility of having students move from traditional to modern to traditional courses in a bewildering succession. It therefore suggested that senior high school principals and mathematics teachers take the lead in doing some coordinated planning with their feeder junior high schools and that any increase in participation for the next year be within such a coordinated plan.

As the Steering Committee had anticipated, in the spring of 1962 there was another tremendous increase in requests from teachers and schools for modernized materials. Another event occurred that spring which was to have significance for the mathematics curriculum: The Board of Education authorized an enlargement of the central office staff to include, among other positions, a curriculum director for science and mathematics. The person who filled this position was new to the system, but he rapidly developed a comprehensive view of the situation in mathematics and a good working relationship with the local teachers who had spearheaded the movement. During the 1962-63 year, working closely with teachers, he found (1) that the number of teachers teaching modernized materials was great enough to make a more systematic approach advisable, and (2) that this same factor suggested that most of the secondary teachers were ready for comprehensive use of the newer materials. These decisions led to the revised program for 1963-64 which was described earlier. (It should be noted that, although the central office staff projected a system-wide pattern, considerable latitude, including the right to experiment with some new courses, was left for principals and teachers.)

In addition to the development of this program, a systematic start was made toward updating the elementary and junior high mathematics programs. New materials were obtained for some classes in every junior high school, and a worktext reflecting the new approach was obtained for all first graders. It was further decided that in 1964-65 such worktexts would be obtained for both first and second grades, thus revising elementary mathematics from the bottom up. This progression continued until the spring of 1966, when the state adopted a new elementary mathematics series which took the desired approach. At the same time, the state adopted materials for secondary mathematics similar to those in use in this school system.

It is interesting to note that we have had more difficulty in maintaining a sense of security among elementary teachers and their students (and the parents of those students) than among junior and senior high teachers. This is perhaps because our junior and senior high mathematics teachers are mathematics majors, whereas our elementary teachers have been trained as generalists to handle self-contained classes. It may also be the consequence of the fact that large numbers of secondary mathematics teachers came into the movement during the voluntary growth period, while elementary teachers were not included until the total movement attained proportions which made it mandatory to set up a systematic approach in grades 1 to 12.

The Future

In spite of our having reached what appears to be the basis for a stable program, we feel that it would be dangerous to relax. Just as we felt in 1959 that our professional obligations dictated a review of the work of the Commission on Mathematics, we now feel that ideas such as those presented by the Cambridge Conference on School Mathematics must be studied, and that decisions must be made as to the meaning of these ideas for us. In addition, we are aware that there is a rapid increase in the amount of advanced placement mathematics, both formal and informal, which many school systems are offering. Experience suggests that what is advanced placement in one decade may well be part of the standard college-preparatory program during the next.

We are interested in an advanced placement program, not just for its own sake at the present moment, but for the implications it may have for the future. We are, therefore, beginning a "five-year plan" which will hopefully enable us at the end of that time to offer elementary analysis as the twelfth-grade course for the average college-preparatory student and to offer advanced placement for superior students. We hope to accomplish the latter by merging the essentials of first-year algebra into the stronger seventh- and eighth-grade programs now in effect, rather than having students "skip" eighth-grade mathematics to take Algebra I. This will enable such students to complete the analysis course in the eleventh grade and be ready for advanced placement or optional honors courses in the twelfth grade. The first group to try this approach is now in the eighth grade.

There are two other areas which concern us as we look to the future. The first is the question of what form an expanding program in general mathematics should take. Should we continue to give major emphasis to mathematical structure as it applies to general mathematics, or should we place increased emphasis on the utilitarian approach? Is it possible to fuse these two approaches? A committee composed of mathematics teachers, vocational education teachers, and lay advisers are to consider this question during the current school year and to attempt to outline an approach which can provide the pleasure of looking at structure and the value of developing applications.

The final area of concern is the broad question of using a more laboratory-oriented approach in our mathematics program. The implications of the computer, as well as those of less exotic equipment, demand attention.

The most significant feature of our story is probably not the current status of our mathematics program. (Many readers will quite surely feel that their own programs are even more advanced.) The most significant feature to me is that we have arrived at our present status through a mixture of human relations and mathematics, which provides a sound basis for ongoing study and revision.

It is significant that the school administration chose to exercise leadership, first by stimulating teachers and second by coordinating the teacher activity which resulted from this stimulation. In addition, even after developing a coordinated plan, the school administration left considerable room for initiative with teachers and principals. Mathematics has not, therefore, become a vehicle for creating opposition between teachers on the one hand and administrators and supervisors on the other. This atmosphere enables us to feel reasonably comfortable with our present situation and ready to work together on further developments.

"What, then, are the factors which seem of greatest significance in improving the quality of a mathematics program? Adequate leadership . . . cooperation and encouragement of the district administrators . . . availability of in-service education . . . and certainly better articulation."

Eugene Develops a Mathematics Program

WENDELL C. HALL

IN 1954 there were 20 elementary schools, four junior highs (grades 7, 8, 9) and one high school in the Eugene, Oregon, public school system. The mathematics program was quite standard and student achievement was about average in mathematics—the average eleventh-grade score was at the fifty-third percentile on the mathematics section of the Iowa Tests of Educational Development. There are at present 31 elementary schools, eight junior highs, and four high schools in Eugene, but the average eleventh-grade score in math now is at the seventy-fifth percentile on the current edition of the Iowa Tests.

Eugene's mathematics program has been updated, of course, but factors such as in-service work with the teaching staff and a revised operational structure have also been extremely important in improving the performance of mathematics students. A description of the procedure followed in this school system may yield suggestions for other systems.

Leadership is an essential, especially when changes are to be made in a somewhat static situation. A well qualified mathe-

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matics educator was employed by Eugene in the fall of 1954 to head the high school mathematics department and to work with student teachers from the local University of Oregon. He also assumed leadership in the district's mathematics program even though such responsibilities were not delineated very precisely. The first major step thus was taken.

There was already a monthly in-service program operating regularly in the district, and the first significant action was the combination into one group of two existing in-service sections of junior high and senior high mathematics teachers. Articulation improved. Discussion identified the main weaknesses of the existing program—nonchallenging seventh- and eighth-grade courses, poor articulation, and vague course objectives, especially for some "special" high school courses—and some positive suggestions were proposed. This initial effort took a year, but a spirit of cooperative effort was established and goals were determined.

Determining Objectives

The second year was spent in developing statements which reflected the philosophy of the mathematics staff and in writing objectives for the various courses already offered. Though the effort was not extremely glamorous, there was general agreement on the need to "get better organized" and most of the working agreements developed at that time are still incorporated in the district program. These assumptions[1] were as follows:

> The secondary mathematics program should meet both the general education and special needs of all students. Therefore such a program must be a comprehensive one, which implies that:
>
> 1. There should be a course geared to their ability level for all students who wish to study mathematics. This course should be easy enough for them to experience some success yet difficult enough to be challenging.
> 2. A student should be allowed to enroll in any course in the program after he has met the prerequisites for the course. In general, a student should be discouraged from taking a course if his chance of succeeding is small.
> 3. The placement of a student in the proper mathematics course should be the responsibility of the student's counsellor, the math-

[1] *Eugene Mathematics Program, Grades 1-12.* School District 4, Lane County, Eugene, Ore., 1967, p. 20.

ematics department chairman, and the student's mathematics teachers.

4. There should be a variety of courses available at each grade level.
5. A comprehensive evaluation program is necessary if the mathematics program is to function well. Data should be collected which will help:
   a. in the placement of students in courses;
   b. determine the overall effectiveness of the total mathematics program;
   c. determine the growth made by students in a course—the basis for reports to parents; and
   d. determine the effectiveness of various approaches to instruction.
6. Remedial courses should be available for students as soon as the need is detected.
7. College preparatory students should be encouraged to study mathematics for at least one year beyond Math 4 (geometry).
8. All students should be required to study mathematics nine of the twelve years of elementary and secondary education. If high school juniors do not show average ninth-grade achievement, they should be required to enroll in an additional mathematics course. [This requirement is a policy established by the District Board of Directors.]
9. The last mathematics course a student studies should be taken near the end of his high school career.

Firm proposals were then prepared for solving some of the problems. Although a few advanced eighth graders had been allowed to take ninth-grade general mathematics, it was proposed that they take algebra instead after covering both seventh- and eighth-grade general topics in grade 7. Junior high principals endorsed this plan after being assured that such "tracking" need not be permanent. This acceleration program was begun in 1957 and was established in all junior highs the following year.

The junior high staffs were capable of teaching the more advanced mathematics, but extra in-service sessions dealing with special problems were held for them the first several years.

It was decided that the accelerated students would follow the existing college-prep sequence, moving geometry downward to grade 9, and that a new, advanced course would be ready for them as seniors, not earlier. A summer workshop program in the school district provided the opportunity to determine the content and develop the guide for this course. An advanced placement course in analytical geometry and calculus was introduced in the two high schools in 1961.

Some Courses Revised

The basic structure of the mathematics program was not altered extensively; most changes were made within the existing structure. A course named "Senior Mathematics" was revised to serve as a review of algebra and geometry topics for students weak in those skills, and was renamed "Advanced General Math." It still provides a "catch-up" course for students not capable of competing immediately in the advanced program. Special units were written in summer workshops by the teaching staff, and continue to be revised periodically.

The remedial mathematics course became the responsibility of a particularly resourceful teacher and it soon took on a new look and stature—with noticeable effect. And the topics for the high school general mathematics course, "Intermediate Mathematics," were rewritten to be less repetitious and of greater interest; business topics such as income taxes and insurance were intentionally delayed until the senior year; problem-solving and applications were stressed.

Besides the new junior high courses for advanced students, another was soon created for those junior high students deficient in computational skills and basic concepts. These students can be identified in the elementary schools, so the course is usually taken by seventh graders. Such assistance should not be delayed.

Simultaneously with these developments in secondary mathematics, an in-service program was begun for interested elementary teachers. It was even more attractive because it satisfied district requirements for periodic "professional advancement." The elementary mathematics program was discussed, the concepts and skills involved were explored quite deeply, and behavioral objectives were written and compiled into a revised course of study by the mathematics leader. It was hoped that dependence upon the text could be lessened. However, the bulkiness of the resulting course of study, the lack of teacher confidence in mathematics, and insufficient planning time largely defeated this hope. Yet this group of teachers became much better educated mathematically and provided substantial leadership in subsequent curriculum revisions. It was a major accomplishment.

The mathematics leader had developed considerable acquaintance with the district's mathematics staff in the first three years through in-service work and also through responsibility for the placement and supervision of student teachers. Such personal contacts and disscussions provided an invaluable opportunity to assess the capabilities of the staff members and the effectiveness of the curriculum.

Suitable Scheduling Stressed

The greater emphasis upon scheduling a student into a suitable group created the request by teachers for a card recording past achievement and other useful data. A card was prepared for use throughout the secondary level, and a similar one still follows the student from teacher to teacher. The practice of sending the grades of seventh-grade students and sophomores to the sixth-grade and ninth-grade teachers respectively began later, and that information has improved course scheduling which, for mathematics, remains largely the responsibility of the teacher.

Three additional events were to affect Eugene's program in the 1956-60 period:

- Oregon was included in the Science Teaching Improvement Program (STIP) of the American Association for the Advancement of Science, from 1956 to 1958.
- The school system was designated an experimental center for School Mathematics Study Group (SMSG) materials.
- The State of Oregon Course of Study was being revised.

Impetus to re-examine existing classroom methods and curriculum content and to become familiar with a greater variety of supplementary materials and visual aids came from STIP. Statewide contact between mathematics educators began, leaders developed, and progress was begun. The experimental SMSG materials stimulated the teachers involved, and ideas and results were shared via in-service sessions; thus, the enthusiasm and capabilities of teachers increased generally. Two staff members were eventually chosen for SMSG writing teams and this liaison proved most advantageous. Reports from newly established National Science Foundation institutes were also informative. The involvement of key district personnel in the revision of the State Course of Study necessitated considerable cogitation about the form of Eugene's program. The effect of this was substantial and the basic "single track with 'sidetrack' modifications" structure evolved at this stage.

Possible course sequences are illustrated by the accompanying diagram. Capable students would continue from M1 to M6 in grades 7 to 12. Some students would be accelerated via the Math 1-2 course in grade 7; many students would take several general or review mathematics courses. A brief description of each course also follows. This material is taken from the *Eugene Mathematics Program, Grades 1-12* referred to previously.

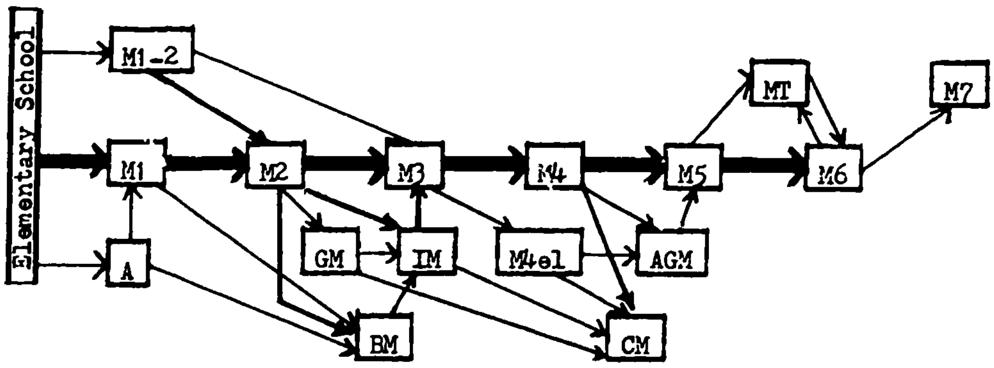

Diagram of Possible Course Sequences

Though the arrows in the diagram indicate most of the sequences available by following "sidetracks" for remedial or other purposes at strategic points, other sequences can be arranged after careful study by teacher, parents, counselor, and student. It is possible for a student to repeat a course; it is permissible to take no mathematics in grade 9 if this seems desirable; it is even possible to accelerate by taking summer school classes of additional work (programed material, double classes), or because of outstanding achievement.

Course Descriptions

M1—Math 1. Fractions and decimals; ratio, proportion and percent; properties of arithmetic numbers; metric geometry; bases other than 10; intuitive algebra; nonmetric geometry; statistical graphs.

M2—Math 2. Positive and negative numbers; properties of real numbers; coordinates in a plane; ratio, proportions and percent; exponents; metric geometry.

M1-2—Math 1 and Math 2. Accelerated. The essentials of Math 1 and Math 2 in a one-year course.

M3—Math 3. First-year algebra

M4—Math 4. Geometry

M4el—Elements of Math 4. An informal treatment of geometry; proof less emphasized than in Math 4.

M5—Math 5. Second-year algebra, some trigonometry.

M6—Math 6. Analysis (pre-calculus).

M7—Math 7. Calculus and analytical geometry, an advanced placement course.

A—Arithmetic. Remedial; normally for seventh graders.

GM—General Math (junior high) and IM—Intermediate Math (high school). Overview of algebra and geometry emphasizing measurement; ratio, proportion and percent; graphs; formulas and equations.

BM—Basic Math. Remedial.

AGM—Advanced General Math. A review of algebra and geometry with introduction of new topics.

MT—Modern Topics. One-half year of modern abstract algebra and one-half year of statistics.

CM—Consumer Math. Problems in family finance and in business. Not remedial. Open only to seniors.

Of the courses listed, the only recent additions are "Elements of Math 4" (M4el) and "Modern Topics" (MT). The former resulted from a study on why students were avoiding Math 4 or were having difficulty with it. Since geometry is considered of value to most students, the emphasis on deductive proof was lessened in favor of intuition and inductive proof, and M4el resulted; it has proved to be effective. The content of Modern Topics was suggested by several mathematics curriculum reports and it provided an alternate study for the noncalculus-bound student.

Articulation of Programs

By 1962 the secondary program was reasonably well developed and contact with the sixth grade was well established so students could be placed accurately in the seventh-grade program. But the proliferation of new mathematics materials and the need for better prepared teachers at the elementary level dictated additional attention for that level. At the same time, a "resource teacher" program for elementary schools in Eugene provided an additional teacher for each building according to the needs of the school. Many of these teachers had been participants in the previously mentioned elementary in-service mathematics class, hence mathematics received substantial attention from them. Early SMSG materials were taught successfully.

In-service classes on SMSG content, primarily sixth-grade level, were taught on television over a four-year period (1961-1965) by three experienced teachers and were supplemented with monthly meetings with the TV instructor for the first several years. Sixth-grade teachers, resource teachers, and others who were interested attended these after-school sessions.

In addition to the SMSG books, several new series were also tried so that at one time six different programs, old and new, existed in the district. As a result of this experimentation, a new elementary textbook series was selected by a committee of teachers; it was instituted in grades 1 and 2 in 1964 and in the remaining elementary grades the following year. Further, intensive in-service training sessions were held throughout the district. These included not only mathematics concepts, teaching techniques and aids, but a question-and-answer period. An adequate number of teacher-leaders was again important for they served as instructors and assistants.

The Eugene school district's extensive workshop program and extended contract schedule enabled teachers to participate in the evaluation of materials, of subject matter, and of the curriculum, and in the preparation of necessary materials. The administration has been most cooperative in implementing requests and recommendations from the teacher participants, which has strengthened the educational program considerably.

Two other innovations in Eugene's mathematics program have proved of value: the Mathematics Advisory Council, which is charged with the success of the district's mathematics program; and the Area Mathematics Coordinator structure, which places responsibility for implementing the mathematics program on four mathematics educators, one for each high school attendance area, rather than on just one supervisor.

Staff Involvement

Eugene's leaders in mathematics believe that group planning, group action and group responsibility are advantageous; hence, the responsibilities usually assumed by a single director of mathematics have been delegated to the Mathematics Advisory Council (MAC). Members selected by the administration always include the high school department chairman, the elementary mathematics consultant, and the area coordinators; other members include representatives from the elementary and secondary principals groups, the elected chairman of mathematics in-service training, and representatives of junior high, high school and elementary teachers.[2] The chairman of the Council is the acknowledged mathematics leader of the school district. Meetings are held monthly to examine, assess, report upon, and discuss the mathematics program. Subcommittees are active in many areas and utilize a large number of teachers in the district.

Significant action taken and policies agreed upon by the Council must be approved by appropriate directors of secondary and elementary education and the district superintendent in order to be official. This is seldom refused; hence, the system is extremely functional and may soon be extended to other subject areas. However, several words of caution must be sounded: teachers innovate and work willingly on their own time if they note personal and educational growth and solution of problems important to them. But if morale and efficiency are to be maintained, additional time and financial renumeration must eventually be provided by the district. Also, such a structure does demand a high degree of cooperation and compatible personalities if it is to work as smoothly as it has in Eugene.

The second noteworthy innovation was instigated by a report from Eugene's superintendent and several Board of Education members after they had observed a regional education structure in Pittsburgh, Pennsylvania. Soon thereafter, for various administrative purposes, the Eugene school district was divided into regions based upon high school attendance areas and the schools located within them. Again, leadership power was diffused rather than concentrated when a structure was established for mathematics. Four area coordinators were appointed by the administration from nominees presented by the MAC. These area coordinators are released at least one hour daily from teaching

[2] Present membership of 20 consists of chairman, Area Coordinators (4), high school department chairmen (4), elementary teachers (3), elementary principal, secondary principal, elementary mathematics consultant, in-service chairman, a high school teacher, and the junior high representatives (3).

duties to work with teachers in grades 1 through 12. They are also given a prorated supervisory increment. The total cost of this arrangement is no greater than if a single full-time person were hired, and here the coordinators maintain contact with the classroom and a broader base of leadership is recognized, two advantages thought to be important and effective.

These area coordinators work cooperatively with the principals and department chairman to implement the approved mathematics program and decisions of the MAC. They consult with teachers on all levels, are responsible for special in-service meetings within their area, assist new teachers especially, and promote more effective articulation throughout the 12 grades. Teachers are selected who have a wide range of experience, at least junior high and senior high, and who are able teachers and leaders. Currently, two are junior high teachers and two are from the senior high level. This, too, has been successful enough to warrant imitation by other departments in the district.

Local Leadership

The repeated emphasis on leadership has had effect. Many of Oregon's leaders in mathematics education are in the Eugene school system; several are quite active in the National Council of Teachers of Mathematics and its projects. As new schools opened, a nucleus for a new mathematics staff was ready from within-district personnel. Mathematics teachers have also been active in other professional associations and activities and have contributed to education on a broad front.

Teachers exercise substantial influence in many areas. They select texts, serve on curriculum and special committees, often cooperate in determining their teaching assignments; they plan and lead in-service groups; and their representatives serve on the Mathematics Advisory Council. The entire structure provides for communication and articulation. Indeed, there is almost too much activity, and it is well that an extensive summer program permits many of the larger problems to be attacked there.

A most effective morale booster has been the assignment of teachers, even new ones, to courses suitable to their interests and training—not just on "seniority" alone. Each secondary teacher can usually anticipate having at least one "good" course and also the experience of working with more difficult students. Hence, most teachers soon attain a wide range of experience (which makes it easier to schedule classes), and the turnover rate is lessened.

How are decisions reached within such a structure? Let me give you an example. A problem which arose because of the accelerated program was the equitable grading of students: It is essential to avoid penalizing students in grades earned just because they are competing with extremely able students rather than with a cross-section. Individual teachers and schools requested a uniform policy. First the MAC established a subcommittee of six members on "Grading Policy." Meetings were held, preliminary reports were submitted to and amended by the MAC after reactions from teachers in building department meetings, and, after a year of study, a final report was made and accepted by the MAC. It was then approved by the administration, and reported to the school board. This policy seems to have functioned satisfactorily according to the Area Coordinators, who confer periodically with teachers in their area.

Problems and Prospects

Some problems still remain. Many proposals are stymied because of the inflexibility of current programs, class schedules, and buildings. Finances are a persistent problem. A higher degree of teacher competence must be achieved. And not enough provision is made yet for the differences in individual students. But the program described has provided solutions to substantial curriculum and articulation problems; further cooperative effort should produce more acceptable solutions, though perhaps slowly.

What, then, are the factors which seem of greatest significance in improving the quality of a mathematics program? Adequate leadership is certainly essential; the cooperation and encouragement of the district administrators are needed to establish a unique, but functional structure; the availability of in-service education and summer workshops are important; better articulation also helps and that is a contribution of such groups as the Mathematics Advisory Council, the mathematics coordinators, and cross-level in-service groups. Teacher morale improves when avenues of communication are clearly open, when teachers have a direct influence upon programs, and when their teaching assignments are determined cooperatively not arbitrarily, and on the basis of ability and interest not just of seniority.

Those factors have been evident or implied from the previous information in this article. But note especially that the basic philosophy of the district provides the framework within which the program develops and that a workable structure does entice the best of leadership and teaching from the available staff.

Many school systems have, without fanfare, been creating attractive and effective opportunities by which slow learners can get somewhere in mathematics. These demonstrate, among other things, that the day of the locally-made curriculum is by no means over.

The Slow Learner—Changing His View of Math

RUTH IRENE HOFFMAN

FEAR of mathematics and a distaste for any computation or for the kind of analytical thinking that typifies mathematics—these qualities characterize the slow learner in mathematics. This fear and distaste are born most frequently of some early fuzzy understanding or lack of understanding and are nurtured in succeeding years by the frustration of attempting to build new understandings on a nonexistent foundation.

The student becomes, in consequence, a slow learner in mathematics or, more properly, a nonlearner beyond a certain level of competence. He finds all mathematics classes deadly dull; he sees little use for the subject. If he is a docile student, he endures quietly and with great boredom the mathematics course he is required to take; otherwise, he rebels and becomes a discipline problem as a protest against his total lack of understanding and interest.

These students are frequently not slow learners in the sense of having low intellectual ability. In fact, many have high IQ's, and the intensity of their negative attitude toward mathematics may well be directly proportional to their intellectual ability.

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Part of the many-phased development of the mathematics program in secondary schools is the burgeoning interest in programs for the low achiever. This interest seems to have sprung up in every part of the country and, surprisingly, the resulting programs are very similar in nature, although developed independently.

There is one national organization which has done much to enable the personnel developing various projects for the slow learner in mathematics to share their materials and techniques with each other. This organization, under the direction of Paul C. Rosenbloom of Teachers College, Columbia, is called CAMP (Concepts and Applications of Mathematics Project). The group is hampered by lack of funds in sharing its work on a broad basis, but some of its members meet annually at the time of the NCTM meetings.

Elements Common to Math Programs

The directions that projects for the slow learner in mathematics share can be related to two aspects of the so-called "modern mathematics" movement:

1. Use of a mathematics laboratory, with all its ramifications including calculators, remote terminals for computers, and flow-charting for problem analysis.
2. Emphasis on the structure of the number system and the beauty and interest of patterns in mathematics, and inclusion of selected topics from number theory, intuitive geometry, and informal topology.

To summarize all aspects as falling in those two caegories is an oversimplification, however, and certainly leaves out important developments of the program.

The initial meeting of CAMP was held in Iowa, partly because of the pioneering work done in the Des Moines Public Schools with the low achiever. The materials produced by this school system are entitled LAMP (Low Achiever Motivational Project). The Des Moines program is centered around a laboratory classroom, and the materials begin with the following excellent definition of a mathematics laboratory.[1]*

* Numbers in brackets refer to sources listed at the end of this article.

Primarily, a mathematics laboratory is a state of mind. It is characterized by a questioning atmosphere and a continuous involvement with problem solving situations. Emphasis is placed upon discovery resulting from student experimentation. The teacher acts as a catalyst in the activity between students and knowledge.

Secondarily, a mathematics laboratory is a physical plant equipped with such material objects as calculators, overhead and opaque projector, filmstrips, movies, tape-recorder, measuring devices, geoboards, solids, graph board, tachistoscope, construction devices, etc. Since a student learns by doing, the lab is designed to give him the objects with which he can do and learn.

The primary goal of the lab approach is to change the student's attitude towards mathematics. Most students have become so embittered by habitual failure that they hate mathematics and everything connected with it. There is little possibility of this student learning mathematics until an attitude change has been effected. It is because of this goal that our approach is different. Some would label our approach as "fun and games," but I am sure that close examination will bring realization that everything in the program is oriented toward the twin goals of attitude change and mathematical improvement. . . .

Jefferson County Project

Mr. Terry Shoemaker, formerly a teacher in the Des Moines Public Schools and one of the original staff which developed these materials, has now been working for two years in the Jefferson County Public Schools near Denver, Colorado. He has continued the work begun in Des Moines in his new assignment. This Jefferson County project[3] is an excellent example of the kinds of programs developing throughout the country and can be used for a case study.

The characteristics of the low achiever upon which Mr. Shoemaker based the development of a special program are those which have been pinpointed by him and others working with this type of student. They are as follows:

- A record of failure in mathematics.
- Achievement scores three or four years below grade level in mathematics.
- Reading and comprehension difficulties in. many cases.
- Quick conclusions formed without due consideration of facts.
- Interest span very short—10 to 15 minutes at any activity.
- High rate of absence.
- Goals based on view of *immediate* future.
- Antisocial behavior exhibited in classroom and school.
- Inability to see practical use of mathematics.

The program consists of some basic procedures, each gauged to meet the needs of students having the foregoing characteristics.

1. *Multiple Activities.* Each day's activity is a self-contained piece of work which can be completed during the class period. It is also immediately evaluated. This requires the preparation of much independent material in the way of work sheets. These work sheets are carefully planned and made available through an easy filing system. The student is also provided with at least two changes of activity during a class period. The activities are local business problems, specially developed drill problems, and many types of exercises including crossnumber puzzles.

2. *Electric Printing Calculators.* One calculator is provided for each two students. The calculator is used to encourage such ideas as decimal point location, estimation, discovery of error patterns, discovering basic interrelationship of fundamental operations, and checking computation which has been done by hand. The students exhibit great interest in using the machine and have pride in mastering its operation.

3. *Local Business Problems.* Local businessmen contributed the most common mathematical problems which they, as employers, faced in their business. The problems were submitted by each businessman on the letterhead stationery of his company and are presented to the student in this form. Since the student recognizes the local firms and knows these are current problems, he feels an interest in solving them.

4. *Flow-Charting.* A natural device to enable students to analyze an algorithm or a process of computation and break it down into its basic parts, or to break down a problem or a situation into its intrinsic operations is the flow-charting technique used for computer programing. In the initial stages, students begin with nonmathematical situations such as "starting a car," "getting ready for school in the morning," or "calling a girl for a date." This is followed by simple algorithms and noncomplex word problems, presenting problem-solving in a new light.

5a. *Mathematics Laboratory Experiments.* Inexpensive and easily assembled physical apparatus is gathered into kits to provide students an opportunity to arrive at mathematical ideas through experiments—comparing, recording, and analyzing to develop mathematical formulas and relations based on physical evidence.

5b. *Laboratory Multi-sensory Aids.* Standard audiovisual equipment is used, but the use is different from that in the usual classroom. A dictating machine with earphones is used, sometimes to provide explanations to those with reading difficulties and sometimes for drill in basic addition and multiplication facts, for commentaries on filmstrips, for related activities. The overhead projector is used for the usual purposes of presenting prepared material, but is also used in conjunction with a copying machine to allow students to present their own flow-charts, graphs, or solutions of problems to the rest of the class. Other equipment includes calculators, scales, slide rules, standard mathematical equipment and models, and much equipment for individual student use.

5c. *Laboratory Involvement Projects.* Many puzzles and kits for individual use are part of the laboratory situation. For lasting interest and complete understanding, students must be *personally* involved in the working of mathematics and mathematics-related problems. Puzzles and games that can be checked out and worked individually or in small groups provide one aspect of the program. These motivational devices must be carefully chosen and the use of them controlled.

The Jefferson County school system has evaluated the work done under this special program and feels that it has aroused the interest and developed the skills of students who were formerly disinterested and unmotivated. The program has been expanded to two additional schools this year, to reach and help more of the low achievers in mathematics.

In using Jefferson County as a case study, it must be emphasized that this is only one of the school systems pioneering in a breakthrough to help the low achiever. Similar developments have taken place in every part of the country. The schools show a great willingness to share their materials and techniques with each other. There are many other schools that should be mentioned.

Levittown and Other Projects

For example, Levittown Public Schools, Levittown, New York, have been doing careful work in the development of the laboratory approach to mathematics as is indicated in the following excerpt from a 1967 report:

> The Levittown, New York, project (now a joint project of six school districts including Greenport, Long Beach, Newburgh, Riverhead and Pearl River) is concerned with developing a laboratory approach in mathematics, as a process of both problem attack and developing sound understanding of concepts and the computational algorisms. It is also concerned with upgrading computational skills for large numbers of our students.
>
> Our initial approach has been the booklet *Flow-Charting the Logical Processes of Mathematics—A Laboratory Approach.* This self-teaching booklet for both student and teacher makes it possible to launch a laboratory approach in mathematics via flow-charting and calculators, the outcomes of which are the ability of students to analyze and flow-chart problems, algorisms and properties, etc., arriving at a final solution via the calculator.[4]

The laboratory approach need not be confined to students of secondary schools; the basic idea of an experimental approach to mathematics is the same at all levels. The Nuffield Mathematics Project has done interesting work with children age five to thirteen, well-summarized in the booklet, *I Do, and I Understand.*[6]

Many of the school systems put together material for these low achievers utilizing to a great extent the methods and materials of existing modern programs. Adaptations for low-achieving *secondary* students have frequently come from modern programs for *elementary* pupils. For example, there is common agreement that students who have difficulty learning should have an opportunity to learn through several senses at a time—seeing, hearing, doing, manipulating. These students have short attention-spans and while needing the security of a daily routine also need interesting variations. They need review, but presented in many different guises; they need frequent reinforcement of skills; they need to develop the skill of analysis through various approaches.

The Discovery Method

Most school programs for the low achiever include modified versions of the discovery method—finding patterns in multiplication tables, developing sequences of numbers, developing magic squares, matrix games, remainder arithmetic, number line relationships, number sentences (possibly with frames), logic games, such as the various games with peg boards, the tower of Hanoi, reconstruction of dissected squares and rectangles, tanagrams, topological puzzles of wires and strings. These uses of the mathematics laboratory approach challenge the student to analyze, to use various of his five senses to arrive at solutions, and to feel his own power to accomplish without the pressure of time or of competition with a group.

Again, as in the modern elementary school programs, basic ideas are approached intuitively through discovery and experiment. The abacus is used to develop an understanding of the structure of the number system, the meaning of place value, and the interrelationship of the various operations. Students almost universally enjoy working with man's earliest computer, and it provides motivation and interest as well as building basic understandings somehow missed in the elementary school.

The geoboard with colored rubber bands develops basic ideas of the unit of length and the unit of area, as well as leading to the discovery of geometric properties of plane figures. Geometric models which can be handled—their vertices and faces counted, their interiors filled with liquid or sand and compared for volume—are interesting developments of the basic concept of metric geometry. Experiments with numbered cubes and with dials having spinners on them are frequently used to develop the basic ideas of chance and probability.

Topics from number theory are included with such familiar techniques as the factor-tree, the sieve of Erastosthenes, and the language and symbolism of sets to find the least common multiple and greatest common divisor. Visual representation in the form of geometric models is used to present the meaning of fractions, equivalent fractions, and the adding and multiplying of fractions. Divisibility tests based on the properties of numbers are explored in the discovery manner, with time given to computational checks based on number properties, such as casting out nines.

Widespread Effort

Most school districts have brought materials together in mimeographed form, since the content comes from various existing programs but is not presented in the order or mode needed for secondary school slow learners. It would be impossible to mention all of the fine materials produced by teacher committees and mathematics supervisors, but some of these are listed at the end of this article.

Some of the other cities in which the public school system operates excellent projects are: Dallas, Texas; Stockton, California; Oklahoma City, Oklahoma; Fresno, California; and Long Beach, California. Projects are also in operation at Pacific University, Fresno, California; Salt Lake Regional Laboratory, Salt Lake City, Utah; and Western New Mexico University, Silver City, New Mexico.

Closely related to programs for the slow learner are special projects such as that in the Los Angeles City Schools. There, working under a state-funded program, educators are developing creative, innovative, and unique approaches in teaching mathematics to average-ability disadvantaged pupils in junior high school who are one or two years below grade level. The Los Angeles schools are using many of the methods, materials, and techniques already mentioned, but guidance services and in-service education are basic parts of their program.

During the summer of 1967, several special programs were held for teachers of the low-achiever in mathematics. Mr. Kenneth Kidd held the first of these in early June at the University of Florida. That program emphasized activities for classroom laboratories involving field work of various types, similarity, measurement, ratio, slide rule, calculator, mathematical games, space perception, and related mathematical ideas. It was followed by other workshops of a similar nature in various parts of the country, but the one at the University of Denver during the last two weeks of June is worth looking at for two reasons in particular:

1. Teachers not only discussed and used laboratory material but "mass-produced" equipment to take back with them to their own schools.
2. Follow-up contact has been maintained with those who attended.

As with most of the institutes and workshops held during the past two years, the program at the University of Denver provided time for teachers to identify problems, to pinpoint the hardest units to teach, and to analyze the characteristics of the low achiever. The emphasis was on the laboratory approach, and the teachers attending had an opportunity to see such a laboratory in use at the Jefferson County schools. They also had a chance to work in such a laboratory at the University of Denver.

The Denver Workshop

The scope of the University of Denver program may be summarized as follows:

1. *Acquaintance with the use of calculating devices.* This required having teachers become familiar with a desk calculator and how students may be introduced to it, not to bypass computation but to reinforce it. The teachers also had an opportunity to experiment with small classroom computers, and to study the possible use of remote terminals (tied in to the University of Denver computer) for developing units in computer-aided instruction for review, drill, reinforcement, and preview.

2. *Training in the use of experiments in the mathematics classroom.* The technique of experimentation and discovery used freely in science classes may well be used to aid understanding of mathematical principles and relationships. The process includes having students develop their own means of recording and analyzing data, measuring and comparing diverse objects, and arriving at conclusions based on their experimentation. These ends are accomplished by assembling kits of needed material, providing a center for the material and simple direction sheets and questions.

The teachers attending the workshop actually did some of the experiments, discussed them, and received many suggestions for other experiments to be used in their own classrooms.

3. *Reviewing and learning new techniques of presentation of units.* This includes a discussion of various approaches—flowcharting, use of business forms from neighborhood firms, and types of short units or exercises for reinforcement of learning.

4. *Actual presentation of basic topics needed by the slow learner in secondary school—for example, addition of fractions.* The teachers studied some of the new techniques and emphases (such as the factor trees) being used in new programs in the lower grades. They also studied techniques used to introduce fundamental ideas (such as the concept of place value, which the

slow learner may not have understood) by means of various manipulative devices; the presentation of remainder arithmetic and its purpose; and various techniques for presenting intuitive algebra and informal geometry.

5. *Experience in using devices that belong in a classroom or school mathematics laboratory.* This includes operations on the abacus; use of cuisenaire rods for reinforcing the concepts of number relations, ratio, and fractions; practice in use of the hundred board, the number line, the geoboard; the use of meaningful logic puzzles; graphing games; and topological puzzles.

6. Perhaps the "different" aspect of this particular workshop was a development of the workshop itself, which was not completely planned ahead. Actually, it had some of the characteristics of an assembly line in which, by much work and careful cooperation, *every piece of equipment (except the calculators), every game, puzzle, and device was made by the participants.* Each member carried back to his school not only a notebook of ideas, but one actual example of everything presented to help stimulate, motivate, and aid the low-achiever in mathematics. Thus, each teacher took back a classroom abacus, a set of cuisenaire rods, a geoboard, a tower of Hanoi, one each of the various puzzles, and so on. (This spontaneous activity of the teachers in producing useful equipment and in sharing with each other provided them with a living example of the meaning and value of "involvement" to any learner.)

7. Another indication of the innovative quality of this workshop was that *the participants requested a follow-up meeting in the fall.* In early November this meeting was held, new ideas were shared, and reports were made on mathematical laboratories in operation or about to go into operation. In addition, materials assembled during the workshop from other centers throughout the country are in constant circulation among the participants.

Summary

The two case studies described in this article—the Jefferson County program for the low-achiever in mathematics at the secondary school level, and the University of Denver workshop for the teachers of these low-achievers—delineate a *single* approach to a teaching program and a *single* training program for teachers. These programs have been fruitful, resulting in great strides in motivating the low-achiever. From informal evaluations of attitude and formal evaluations of achievement of students affected by these two programs, there is no doubt that an advance has been made toward changing the attitude of the low-achiever in mathematics.

Common elements in programs for the low-achiever throughout the country are as follows:

1. A mathematics laboratory—whether it be a center for the school, a formal laboratory for the use of a few selected classes, or a classroom laboratory.
2. The use of calculators to help the student find his pattern of error in computation and to enable him to get past simple computational blocks to basic mathematical understandings.
3. A regulated program, with a pattern of activities for security but with a change of activities to accommodate the short attention-span of the slow learners and the unit-a-day pattern for the satisfaction of a task completed and evaluated on the spot.
4. Provision for reinforcement of early basic concepts, which may be weak, utilizing the method and techniques of the more modern programs in mathematics—the exploration of the structure of the number system, the experimentation and discovery of patterns and their utilization.
5. The use of many manipulative devices, such as the abacus, cuisenaire rods, geoboards, etc.
6. The proper and controlled use of games, puzzles, and other motivational techniques.
7. Use, where possible, of remote terminals tied into computers for computer-aided instruction units.

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Providing suitable instruction for mathematically talented young people is a responsibility to be shared by everyone from the classroom teacher to the leaders of national curriculum projects. Here are ways in which educators at various levels can exercise this responsibility.

How Provide for the Mathematically Talented?

JULIUS H. HLAVATY
HARRY D. RUDERMAN

AS EDUCATORS it behooves us to try to anticipate and provide for the future needs of both the individual and the society in which he lives. The explosive growth of technology with the concomitant need for mathematicians compels us to look more closely at what we are doing and how we are preparing to meet future needs. In particular, are we preparing to meet the future needs for mathematicians?

Clearly, the future mathematician, whether in pure or applied mathematics, will be drawn from the mathematically talented. In either case, history has taught us that the mathematician is most creative in his twenties and thirties. In order to reach the mathematical borders of knowledge early and expand them, the mathematically talented must move surely and rapidly up the mathematical ladder.

Very often the mathematically talented student is discouraged from pursuing mathematics because he finds the pace slow,

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Julius H. Hlavaty will serve as president of the National Council of Teachers of Mathematics during 1968-70. Harry D. Ruderman is chairman of the department of mathematics at Hunter College High School.

tedious, and boring. The mathematics that he is asked to study is too easy, providing hardly any challenge. His teacher insists that he move at the same rate as and do the same work as all the other students; this he cannot understand. Why should this talented student be forced to move at a snail's pace if he can gallop? Why shouldn't he be challenged to his full capacity? Why shouldn't he be given the opportunity to move ahead as rapidly as his potential permits?

The mathematically talented student is a rare commodity. He should be coddled, nurtured, and encouraged to grow as strong and capable as possible. Our society will be rewarded many times over if we begin to show the right kind of concern for his growth.

What Can the Teacher Do?

There are numerous ways in which the individual teacher can challenge, excite, and encourage the mathematically talented. Here are some of them.

1. The teacher should try to ask the right question at the right time.

What pattern do you recognize here?

How can you be sure that this pattern will continue?

What is another way to figure this out?

Which way do you prefer? Why?

What would happen if . . .

- the conditions of the problem were changed to . . . ?
- the converse were asked, namely, . . . ?
- these limitations were removed, namely, that . . . ?
- these additional conditions were imposed, namely, that . . . ?
- you had to depend on just the information you now carry along and without any text being available?
- you had to convince someone who is not well informed?
- you had to provide the underlying logic for this solution of the problem?

What is a general class of problems of which this is a special case?

What is a general solution for this class?

Justify your procedure for obtaining a solution.

What kind of problem does this remind you of? Why?

What further explorations of this problem would you like to suggest?

A few specific questions might be:

- How many whole number solutions are there to the equation: $x + y = 8$ if solutions as (2,6) and (6,2) are regarded as the same? as different?
- Suppose that we had the equation: $x + y + z = 8$ and we asked the same questions as in the preceding example?
- What kind of geometry would we obtain if instead of having a straight edge available, we had a fixed circle or a fixed equilateral triangle or a fixed square?
- How can you prove that there is no largest rational number that is less than $\sqrt{2}$? Generalize.
- Must one invert when dividing fractional numbers?

2. The teacher could include optional honor problems in homework assignments. Such problems should not be simply longer and harder problems, but rather problems that lead to new ideas and concepts. The challenge might require a bit of originality to resolve.

3. The teacher could include optional bonus questions on regular tests. Just as for honor problems on assignments, the bonus questions should be challenging and lead to new concepts. For example, a challenging question for fifth graders might be to try to obtain three odd numbers that total 30. For a tenth-year geometry class the bonus might be to prove a converse of the theorem that the opposite angles of an inscribed quadrilateral are supplementary. For eleventh- and twelfth-year classes the following equations are nonroutine and challenging: $x^3 = 2^x$ and $x + 10 = 2^x$.

4. The teacher could have available a class mathematics library for students to use, borrow from, or browse in in their spare time. The NCTM has a publication, *The High School Mathematics Library,* listing many excellent mathematics books. The class library should include copies of the 27th and 28th *Yearbooks* of the National Council of Teachers of Mathematics, which deal with enrichment. The library should also include the collection of SMSG monographs called the New Mathematical Library, as well as the series published by D. C. Heath, *Topics in Mathematics,* containing translations from the Russian.

5. The teacher could encourage the talented youngster to report to the class on some of his own discoveries or on some of

his outside readings. Such reports often inspire others to extend themselves and to try to be creative as well as to read outside material.

6. The teacher could encourage the construction of mathematical models and machines. Among the kind of machines talented youngsters have constructed are logic machines, both analogue and digital computers, and slide rules.

7. The teacher could make himself available for consultation and discussion. Just giving the talented youngster an opportunity to talk about his own ideas can be most productive. Such youngsters need listening posts.

8. There are exciting films on the market that can arouse much mathematical thought and interest; for example, the film, "Mr. Simplex Saves the Aspidistra."

9. The teacher could take interested pupils to a computer center.

10. The teacher could encourage talented youngsters to participate in math contests and math fairs; to write up their discoveries for the school paper or the *Mathematics Student Journal;* to subscribe to mathematics journals such as *Mathematics Student Journal, Mathematical Recreations, Mathematics Magazine, Mathematical Log,* and others. Some teachers encourage their better students to become members of the National Council of Teachers of Mathematics, as well as of local and state organizations.

11. The teacher could encourage talented youngsters to help others either on a tutorial basis or in regular mathematics help classes.

12. Perhaps the best kind of perception is to know when "to get out of the way." Only too often the teacher, instead of accelerating progress, interferes with it.

What Can the School Do?

The school can play an essential role in caring for the mathematically talented student. Among many posibilities are the following.

1. The school could provide for homogeneous grouping of students and could select the best qualified teacher to handle the most talented classes. It should be common knowledge that teaching the most talented is *not* a sinecure if the right kind of teaching job is to be done; the challenge to the teacher is indeed a great one. Many more hours of preparation are needed if the lessons are to be demanding and productive. This kind of teacher should be able to distinguish significant discussions and topics from trivialities and time-wasters.

2. The school should recognize the additional obligations and responsibilities such teachers have

- by providing time for consultations with students, for holding mathematics clubs, for holding contests, for running math teams, for conducting math fairs, for arranging visits to computing centers;
- by providing funds to travel to professional meetings and conferences;
- by encouraging self-improvement through attendance at in-service training classes, university courses, and summer institutes, and through sharing experiences and ideas among staff members.

3. The school could invite prominent outside speakers who can encourage and excite mathematically talented students to greater effort.

4. The school could provide adequate library facilities. Very often all one needs to do with a mathematically talented student is release him in a library which has a good collection of mathematics books. Advances are always being made in mathematics in spite of many misconceptions, so it is very important to keep the mathematics section up-to-date.

5. Some schools have weekly seminars for talented mathematics students, who come together and share their thoughts and experiences under the careful guidance of a mathematics expert. They are encouraged to explore areas which interest them.

6. Some schools run their own mathematical contests and fairs. These are arranged on different levels so that the less mature students are not competing with the more mature.

7. Schools could maintain a mathematical bulletin board which provides:

- information on matters such as the next invited speaker and his topic; how to apply for a summer institute in mathematics; how to apply for a Saturday morning course in mathematics; the nature of the next TV mathematics program; how the school mathematics team is doing; how to participate in various mathematics activites, including teams, clubs, contests, fairs;
- a weekly challenge in the form of a problem or possibly different problems for different levels;
- clippings from various magazines and newspapers bearing on mathematics;
- contributions by students in the form of challenges and solutions, as well as discoveries.

8. Schools could provide an advanced placement course in mathematics.

9. Schools could provide a computer course.

10. Schools could provide for contacts with neighboring schools, colleges, and universities. Often provision can be made for a talented mathematics student to attend courses at a college and to obtain help and guidance from a professor of mathematics. Some colleges have computer facilities that can be made available to the talented student.

11. Assembly programs can be used to expand the horizons of information regarding mathematics and its career possibilities. Invited speakers can be obtained from schools, colleges, industry, and government.

12. The school could publish a mathematics newspaper or magazine with articles prepared by students. Articles could contain novel solutions to problems, original discoveries, biographies of mathematicians, problems open for solution, famous unsolved problems.

13. Schools could organize clubs concerned with computer mathematics; recreational activities, chess, checkers, Go; the slide rule; construction of models and mathematical machines; the math team and contest preparation.

What Can a School System Do?

A school district can and should make appropriate provisions for its mathematically talented students. At the district level many things can be done that would not ordinarily be done at a local level. Here are some.

1. Attract highly qualified mathematics teachers to the district by offering inducements in the line of time off, opportunity for self-growth, opportunity for increased salary, and opportunity for advancement.

2. Improve the quality of present teachers

- by planning appropriate in-service training courses;
- by encouraging teachers to take these courses;
- by advertising all courses—in-service, college, summer institute, Saturday—that will improve the quality of teaching;
- by holding meetings and conferences at which teachers are given the opportunity to share experiences and problems and suggest alternatives not in practice—a demonstration lesson by a master teacher always provides an exciting session, especially if followed by open discussion.

3. Provide for mathematics contests and fairs with much publicity to arouse excitement and enthusiasm.

4. Use local radio and TV facilities to publicize courses, give courses, praise winners of contests and their teachers. These facilities may well be used to inform talented students in single-session talks about selected topics in various areas of mathematics. Many such talks are on film: "Mathematical Induction," "Mr. Simplex Saves the Aspidistra."

5. Encourage visitation of teachers between schools. Teachers can learn lots from each other if given the opportunity.

6. For large school districts, have special schools in which the concentration of mathematically talented is increased. Such special schools permit more electives in mathematics. Just the possibility of having another student with similar interests to talk to has proved of tremendous value in increasing the amount of effort put forth by the mathematically talented.

7. Encourage schools to experiment with new programs, new techniques, and new materials. This may include the use of programed materials, films, visuals, overhead projector, etc.

8. Provide for:

- a central film library

- an extensive mathematics library in each school and in the district library
- a resource center where students and teachers may go to obtain assistance
- wide publicity for outstanding work from students as well as teachers
- a mathematics speakers bureau to make it easy for a teacher to obtain a speaker who will arouse and excite interest in mathematics.

The procedures and techniques discussed up to this point are intended to provide a richer experience for the gifted in any curriculum and in any school organization, but in actual practice they can provide a better program of mathematics education for all students.

Curricular Provisions

Within the past 10 or 15 years, some curricular programs in mathematics have explored a new dimension. These programs have, for the most part, been directed towards the more capable students. The Report of the Commission on Mathematics was specifically prepared for the college-bound high school student. The University of Illinois Committee on School Mathematics, the School Mathematics Study Group, the University of Maryland Mathematics Project, the Ball State College Program, and others proposed new content and new organization of content. Widely adopted, these have been used principally with talented students.

In 1963, the Cambridge Conference on School Mathematics opened a new level of curriculum discussion, for the elementary as well as the high school. Assuming that the relatively modest proposals of the earlier reform groups are realized, what should be the mathematical education of the able, as well as of the general high school population?

In Western Europe, with its selective secondary education, bold and dramatically different programs have already been embodied in experimental syllabi and textbooks. Secondary school pupils in Western Europe are comparable to those we in the United States call the mathematically talented.

Many current experiments here, inspired by the Cambridge Conference, are being directed initially to the talented. Among these are: The School Mathematics Study Group—Second Round (Director, E. G. Begle), Secondary School Mathematics Curriculum Improvement Study (Director, H. F. Fehr), and the Comprehensive School Mathematics Program (Director, B. Kaufman).

Advanced Placement

One of the earliest of the programs to provide both enrichment and acceleration for the talented began in the early 1950's. This program, under the Ford Foundation, selected particularly able eleventh-year students and enrolled them as college freshmen. Experience soon showed that while these premature college students could cope with undergraduate and even graduate courses in mathematics, they were missing something in not completing their high school careers and were handicapped in trying to cope with both a college program and a college environment.

In 1954, the College Entrance Examination Board took over a study begun under sponsorship of the Carnegie Foundation to investigate the possibilities of allowing high school students to do college-level work at their own high schools in their senior year, receiving advanced placement and even advanced credit in various subjects. (The group that conducted this study was the Mathematics Committee of the School and College Study of Admission with Advanced Standing.) This advanced placement program today enjoys almost universal acceptance by both secondary schools and colleges. In mathematics, its culmination is a test in calculus administered by the Educational Testing Service. Outstanding performers in this test receive, in many colleges and universities, college credit and certainly advanced placement in the undergraduate mathematics program.

In order to take advantage of this possibility of earning advanced standing or credit, the high-school student must have completed the traditional secondary school mathematics program before the senior year, in order to leave that year free for the full-year course in calculus and related analytic geometry which constitutes the advanced placement course. This the student can do by starting mathematics in the eighth grade, by accelerating his study of traditional content, by devoting additional time to mathematics in grades 9 to 11, by benefiting from a content-reorganization in those grades, or by a combination of these methods.

Secondary schools with strong programs and qualified teachers (often with the assistance of summer institutes on advanced placement teaching) have been able to send large numbers of talented students into colleges and universities with a substantial head start on a major in mathematics.

Other Possibilities

The twelfth year is of course a critical one in education. In the past decade, it has become even more potentially traumatic. First, there is the ever-increasing pressure of college admission. Second, there is "senior let-down," familiar to every high-school teacher as a hiatus arising because the vital decisions on college admission have been made early in the year and the senior thereafter merely marks time. Third, the reform programs in mathematics bring students to their senior year with a wide variety of backgrounds in mathematics. Fourth, methods of instruction in secondary schools, differing sharply from those at college level, have in general not prepared students for the transition to undergraduate first-year programs.

The needs of the most talented segment of the high school twelfth year are probably met adequately by the Advanced Placement Program, particularly since its mathematics content—the calculus—provides a genuine transition to college. There are students, however, who despite enriched experience in high school mathematics either do not want or are not capable of coping with a college-level calculus course. For them, a number of alternates have been developed, at the school, district, or state level. The Commission on Mathematics itself has made a number of suggestions for half-year and full-year courses.

Perhaps the most promising among these proposals is a course in probability and statistics. Even severe critics of changes in school mathematics concede the transcendent importance of this subject for our time. The Commission on Mathematics even prepared a text for Introductory Probability and Statistical Inference, one now widely used by both high schools and colleges. There are a number of commercially produced texts to meet the needs of this area.

Among other proposals being tried are:

Elementary Functions
Linear Algebra
Introduction to Matrix Algebra
Linear Programing
Computer Mathematics
Analytic Geometry

Finally, schools have experimented with general courses at a higher level. These courses frequently consist of a series of units having little if any logical or organic interrelation. The intent is to open doors, to indicate vistas in higher mathematics, not only for those who will pursue mathematics or science but also for those who will enter such fields as the social sciences, which increasingly demand growing amounts of mathematical competence. One model for such a course is the Mathematics 12 X Program, developed by the State Department of Education of the State of New York.

"Secondary school personnel may be astonished to find that the rapid evolution of mathematics programs that has characterized the last decade of secondary school mathematics is also affecting college mathematics programs. . . . These changes are also seriously needed. . . ."

Evolution in College Mathematics

BRUCE E. MESERVE

A WIDE variety of college mathematics programs are evolving under such pressures as

1. the new roles assumed by colleges and new types of colleges that are springing up in our efforts to meet the explosions of knowledge and student population;
2. the technological changes that affect methods of instruction;
3. the increased mathematical maturity of many of the students who are entering college; and
4. the considered recommendations of teams of mathematicians on panels of the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America.

More than half of our high school graduates continue their education. In order to serve this majority of our students effectively at the secondary school level, we need a general understanding of the programs that they will encounter as they continue their education.

The increasing numbers of young people entering colleges have led to well-known admissions problems. The procedures evolving for handling these students are affecting the mathe-

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matics programs. Massive universities are developing, often with graduate assistants teaching most of the lower-division courses. Many liberal arts colleges are acquiring an increasing degree of specialization in their programs. Conversely, many colleges with specialized programs are trying to liberalize the training obtained by their students.

After World War II we were particularly conscious of the rapid increase in the sizes of existing four-year colleges. Today the most striking phenomenon is the rapid growth of the number of junior colleges and two-year community colleges. Nearly one-third of the lower-division college students engaged in associate or baccalaureate degree programs are in two-year colleges.[1] For at least three years, new two-year colleges have been coming into existence at the rate of about one a week.

The Two-Year Colleges

The diversity among two-year colleges is very great and makes a precise characterization of them impossible. However, the trend is unmistakably toward the use of two-year colleges to aid in providing opportunities for college education for all young people who can profit from it. These new colleges are unhampered by traditions. There are very few preconceived notions of what is, or is not, college subject matter. The emphasis is upon the satisfaction of local educational needs.

Most of the students who enroll in a degree program in a two-year college plan to transfer to a four-year college; slightly less than half of these students actually do transfer. The mathematics programs of the transfer students are patterned after those in the four-year colleges, usually with a greater opportunity for students with weak mathematical backgrounds to remove these deficiencies. Most of these students take mathematics at a pre-calculus level of maturity.

This last statement means that the majority of degree students in two-year colleges are studying mathematical concepts that they could have studied in high school if they had been ready to do

[1] *The Junior College and Education in the Sciences.* Report of the National Science Foundation to the Subcommittee on Science, Research, and Development of the Committee on Science and Astronautics, U.S. House of Representatives, Ninetieth Congress, First Session, Serial H. Washington, D.C., Superintendent of Documents, U.S. Government Printing Office, 1967.

so. For most of these students the first year or two of college is essentially a continuation of high school. Community two-year colleges thus support the belief that society can provide this level of education much more efficiently in the student's home community than elsewhere.

One new role of colleges is obvious when we consider the large numbers of students who take isolated courses, who are enrolled in 'terminal" programs, or who are in adult "continuing education" programs in two-year colleges. These students often outnumber the potential transfer students at a given college. The mathematical content of courses for these students is frequently at the upper elementary school level. The critical need for more effective procedures for meeting the mathematical requirements of these students is beginning to receive some attention from the National Council of Teachers of Mathematics and the CUPM.

Technical colleges and institutes are frequently thought of as two-year colleges. The growth of these colleges has been relatively slow.

Throughout all colleges there is an increased recognition of the need to educate young people along general lines with an emphasis upon principles on which the student can continue to build throughout his career. This emphasis is based in part upon the mobility of modern life, with most people changing the nature of their work three to five times during their working careers.

Rudolph L. Heider[2] summarized the ways in which junior college science teachers must prepare to meet the challenge of the 1970-1980's with four statements:

1. [We must learn] how best to educate vast numbers of students of widely differing backgrounds; especially to find meaningful ways to help the less gifted student achieve his maximum potential.
2. Since we will not be judged by our scientific research but rather by the skill and excellence of our teaching, we must continually look for new ideas in education and keep an open-mindedness toward testing and using any procedures that have promise toward increasing the efficiency of learning. We shall need to do educational research.
3. We need to work more closely with industry and the scientific and engineering societies to develop a meaningful course of study

[2] Rudolph L. Heider. "Meeting the Challenge I." *Science Education in the Junior College*. National Science Teachers Association, 1966, pp. 4-9.

for the less gifted but scientifically oriented student in our two-year technology programs.

4. Finally, we cannot much longer look at education as a means of making individuals more efficient in the art of production. We are forced to become concerned with making all individuals broadly educated persons, each to his maximum potential.

The mathematics programs of these two-year college students encounter the same problems as the science programs. These programs must reflect this new role for colleges in order to be useful, and must differ noticeably from the programs in four-year liberal arts colleges.

Many small four-year colleges, frequently those that are at an early stage in their evolution from normal schools for elementary school teachers, have problems very similar to those of the two-year colleges. One major factor in this similarity is the admissions policy. Most two-year colleges have an "open-door" policy; many small four-year colleges have only nominal admissions requirements.

Traditionally the concern for articulation between secondary schools and colleges has been concerned primarily with students entering four-year colleges and universities. This is still a basic concern, but we should not disregard the needs of the large numbers of our students who enter two-year colleges either in degree programs or in terminal programs.

There is a tendency for both two-year and four-year institutions to become large multi-purpose institutions. For example, one-fourth of all colleges and universities now enroll about three-fourths of the full-time college students. These multi-purpose institutions offer a wide variety of programs. Most of these programs include a mathematics requirement either in a general education context or in preparation for intensive work in the student's major field. The mathematics courses to meet general education requirements deserve, and need, much more serious consideration from the mathematical community than they have received to date.

Technological Changes

The foremost te. .nological changes are made possible by the development of electronic computers. Administrative uses of computers are receiving rapid acceptance. Computation centers serving all aspects of the college community are developing rapidly. The availability of time-sharing procedures is making classroom use of computational facilities both educationally and financially feasible.

From the point of view of mathematicians, computers evolved as an adjunct of numerical analysis. The role of computers has now shifted so that computer science, often combined with information processing, is becoming a separate department in many universities. Computer languages have evolved so that numerical analysis is no longer a prerequisite for using a computer and, indeed, effective use of a computer can be made by students from grade 7 up.

Several university programs now require that all students be able to use electronic computers. An introduction to the use of computers is already being provided in many high schools and nearly all of the major colleges and universities. The use of computers as an integral part of a variety of university courses is becoming well established. The development of calculus and other college mathematics courses making effective use of computers has begun.

Isolated instances of similar developments may be found at the secondary school level. Students who acquire a working introduction to algorithmic languages while in high school gain an opportunity for greater insight into both their high school and their college mathematics. Most high school students strive to learn patterns and generalizations. In order to program problems for a computer one must use patterns and general cases. In this sense the use of electronic computers may be considered as a "next phase" in the student's mathematical development.

One emerging use of computers is in conjunction with programed instructional materials to provide computer assisted instruction. A few colleges such as Oakland Community College and Oklahoma Christian College are attempting to minimize formal course work and to individualize instruction through the use of individual study booths, tape recorders, films, filmstrips, and programed materials. Relative to mathematics, such aids are often visualized as part of a mathematics laboratory approach, in which students work primarily on their own at programed tasks with tutors readily available for assistance and moral support.

Increasing use of automated procedures to provide a basic education in a number of skills may be expected, especially in institutions that do not have a strong intellectual tradition. Specific examples of uses of technological aids in teaching and various plans for the organization of instruction may be found in B. Lamar Johnson's report of an exploratory survey of the utilization of junior college faculty services.[3]

The development of outstanding films and filmstrips has been slow at the college level, as it has at other levels. However, several very promising films have been developed by the Committee on Educational Media of the Mathematical Association of America and by the College Geometry Project of the University of Minnesota.

Advanced Placement

The normal expectation now is that students who expect to make extensive use of mathematics in their major fields will be able to start their college mathematics with a course in calculus (possibly with analytic geometry). This expectation provides the basis for the normal goal of college-preparatory sequences of mathematics courses for most high school students. The fact[4] that essentially one-half of the freshmen who take mathematics in four-year colleges take mathematics courses at a pre-calculus level while the other half start with calculus or courses in which calculus is used indicates that this goal is realistic and that high schools have achieved success in attaining this goal for most of their students for whom the goal is appropriate.

Many secondary school mathematics programs provide opportunities for gifted students to study a year of mathematics beyond the normal preparation for calculus. When only half a year is available, introductory probability with statistical applications and introduction to modern algebra have been recommended.[5] For a variety of reasons, courses based upon these topics have not, in general, received recognition by the colleges. The trend

[3] B. Lamar Johnson. *Islands of Innovation.* Occasional Report Number 6, Junior College Leadership Program, School of Education, University of California, Los Angeles, 1964, pp. 33-58.

[4] A preliminary finding from the Survey of Undergraduate Programs in the Mathematical Sciences, 1965-66, conducted by the Conference Board of the Mathematical Sciences under a grant from the Ford Foundation.

[5] Report of the Commission on Mathematics. *Program for College Preparatory Mathematics.* College Entrance Examination Board, 1959.

at the secondary school level to combine such topics into a course with an emphasis upon strengthening the student's preparation for calculus and broadening his understanding of mathematics properly reflects the attitudes of most colleges. Julius Hlavaty discusses in detail various provisions for these able high school students in another paper in this BULLETIN.

Approximately one percent of the students entering college have completed a calculus course in high school.[6] Procedures are currently evolving for providing these students with advanced placement for either one semester or one year of calculus. Many colleges provide honors programs for these capable students. The honors program at Dartmouth College was described by J. Laurie Snell at a conference in 1961.[7]

Recommended Programs

The National Science Foundation is supporting commissions in eight areas including the Committee on the Undergraduate Program in Mathematics (CUPM). According to *The Junior College and Education in the Sciences* already cited:

> The specific objectives of these commissions are: (1) to serve as a bridge between research and the college curriculum; (2) to accelerate the rate of change toward improvement of undergraduate instruction in the respective fields; (3) to interest senior professional (especially research) personnel and able younger men in teaching problems; (4) to encourage material experimentation with the curriculum; and (5) in fields where problems are numerous, to establish priorities, and generate a sense of direction.

The CUPM was organized by the Mathematical Association of America in January 1953, was funded in its early years by the Ford Foundation, and has received support from the National Science Foundation since June 1960. In its early years the CUPM was concerned with a universal freshman course[8] which "would contain all of the theory prerequisite to any sophomore course in mathematics, covering this, with appropriate exercises and

[6] *A General Curriculum in Mathematics for Colleges* (GCMC). A report to the Mathematical Association of America from the Committee on the Undergraduate Program in Mathematics, 1965, p. 20.

[7] Robert W. Ritchie, editor. *New Directions in Mathematics.* Englewood Cliffs, N.J.: Prentice-Hall, 1963, pp. 44-50.

[8] W. L. Duren, Jr., C. V. Newsom, G. B. Price, A. L. Putnam, and A. W. Tucker. "Report of the Committee on the Undergraduate Mathematical Program." *The American Mathematical Monthly.* Vol. 62, August-September 1955.

problem solving, in three hours per week for a year." The students who completed this universal course were expected to enter a classical calculus course, enter a course in mathematics for social studies, or terminate their mathematical training. The recent activities of CUPM are reflected in their publications, a number of which are listed at the end of this article.

Several panels of CUPM are currently active. The impact of the CUPM recommendations continues to grow. Details of the recommendations may be found in the reports. Practically all students who will be making extensive use of mathematics in their special fields will be required to study linear algebra, probability, and statistics in addition to their work in calculus.

Individual copies of most of the CUPM publications are available free of charge from the Committee on the Undergraduate Program in Mathematics, P.O. Box 1024, Berkeley, California 94701.

Conclusion

Secondary school personnel may be astonished to find that the rapid evolution of mathematics programs that has characterized the last decade of secondary school mathematics is also affecting college mathematics programs. Such is indeed the case in the traditional sequences of college mathematics courses. These changes are also seriously needed in and are starting to appear in the two-year colleges and the college programs for the many students who would not have gone to college a few years ago.

CUPM Publications

Recommendations for the Training of Teachers of Mathematics. January 1961, revised August 1964 and December 1966.

Course Guides for the Training of Teachers of Junior High School and High School Mathematics. June 1961.

Five Conferences on the Training of Mathematics Teachers. Report number 1, September 1961.

The Production of Mathematics Ph.D.'s in the United States. Report number 3, October 1961.

A Catalogue Survey of College Mathematics Courses. Report number 4 by Frederick Mosteller, Keewhan Choi and Joseph Sedransk, December 1961.

Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists. January 1962, revised January 1967.

Ten Conferences on the Training of Teachers of Elementary School Mathematics. Report number 7, February 1963.

Mathematics Text Materials for the Undergraduate Preparation of Elementary School Teachers. February 1963, revised 1965.

Preliminary Recommendations for Pregraduate Preparation of Research Mathematicians. May 1963, revised November 1965 under the title of *Preparation for Graduate Study in Mathematics.*

Tentative Recommendations for the Undergraduate Mathematics Program of Students in the Biological, Management, and Social Sciences. January 1964.

Ten Conferences on the Training of Teachers of Elementary School Mathematics. Report number 9, April 1964.

Recommendations on the Undergraduate Mathematics Program for Work in Computing. May 1964. (Note: A more recent and extensive report from the ACM Curriculum Committee on Computer Science may be found in the March 1968 issue of *Communications of the ACM.*)

Course Guides for the Training of Teachers of Elementary School Mathematics. Fourth draft, 1964.

CUPM Basic Library List. 1965

Teacher Training Supplement to the Basic Library List. 1965.

Mathematical Engineering, a Five Year Program. October 1966.

A Curriculum in Applied Mathematics. 1966.

Calculus and the Computer Revolution. CUPM Monograph by R. W. Hamming, 1966.

The Numerical Integration of Ordinary Differential Equations. CUPM Monograph by T. E. Hull, 1966.

Qualifications for a College Faculty in Mathematics. January 1967.

Forty-one Conferences on the Training of Teachers of Elementary School Mathematics. Report number 15, 1968.

"Instead of having separate mathematics and science classes trying to keep pace with each other, we want a single class devoted to both subjects and emphasizing at any particular time whichever discipline seems appropriate."

The Interface of Science and Mathematics

ANDREW GLEASON

A THREE-WEEK brainstorming session with 25 assorted scientists is an exciting experience. The idea-flow rate was tremendous and the differences in point of view were a revelation in themselves. I am a mathematician and accustomed to seeing things in rather theoretical terms. Many of the participants were experimentalists, however, whose primary reaction to almost any idea was, "How can I design an experiment to illustrate that point?"

The Conference itself was a marathon of plenary discussions, small-group discussions, private discussions, and report-writing seasoned by numerous efforts to build things that actually work. When it was over we left with a thick stack of notes, reports, and position papers, out of which we are now trying to edit a formal report. What follows is my personal report on a symposium held last summer at Pine Manor Junior College by the Cambridge Conference on School Mathematics to discuss the interface between science and mathematics.

Obviously it was impossible in such a brief time to cover a topic as broad as this and for a range of school grades from kindergarten through the twelfth. There were aspects of the

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problem we touched only lightly, others that we deliberately chose to ignore. No doubt there remain many we didn't even think of. Still, a conference of this type has, I believe, much to contribute to education, as long as it is accepted for what it was, namely, an idea-generating session.

No one at the conference felt that he was making a definite plan. Everyone realized that we were not at the blueprint stage. The purpose of the conference will have been accomplished only if the ideas expressed there are circulated throughout the educational system, discussed, reshaped, replaced where appropriate, and finally put into practice.

Not for the Elite Only

There were two major agreements reached at last summer's conference. The first, although extremely important, was easily arrived at. We agreed that education in science and mathematics was not to be thought of in elitist terms. We were not trying to "beef-up" the curriculum in an effort to see how fast we could force-feed our scientifically talented youngsters. Quite the contrary—science and mathematics have become such an integral part of our civilization that it is essential to make them meaningful to every school child.

A remarkably large number of people seem to oscillate between blind acceptance and emotional rejection of anything that sounds vaguely quantitative. This is clearly a failure of our educational system. Our mathematics program seems to have created more hostility than knowledge. In elementary school our science programs have been anemic, and in high school they have been largely a matter of memorization. These programs have failed, not only for students at the lower ability levels but also for many college-bound students, as anyone who has tried to teach a college course in science for the nonscientist will testify.

We cannot afford to have, as we now do, such a large proportion of our citizenry feeling confused or threatened by everything scientific or mathematical. We must develop programs in mathematics and general science which will carry the great majority of students to a reasonable level of competence and will provide a suitable basis for more specialized work beginning in high school or for college-level programs in general science.

Integration of Math and Science Study

Our second major agreement was to think not in terms of coordinating mathematics and science instruction but in terms of integrating them. Instead of having separate mathematics and science classes trying to keep pace with each other, we want a single class devoted to both subjects and emphasizing at any particular time whichever discipline seems appropriate. This is a significant decision and one not to be taken lightly.

Science and mathematics are much like a married couple who can't get along with each other, but can't get along without each other either. Modern science is inextricably bound up with mathematics, while mathematics owes many of its fundamental ideas to science. Certainly neither subject would have advanced to its present state without the other. But, like men and women, the two subjects call forth quite different views of the world; those differing points of view have often brought their proponents into conflict.

It has often been argued that the philosophical differences between science and mathematics preclude their genuine integration. We cannot gloss over the differences; they are quite real. But the points of view embodied in science and mathematics are complementary, and this is the essential point. Now that science is becoming a major subject in elementary school it is important to plan the curriculum so as to emphasize this complementarity. Otherwise, our children will probably be unable to see the relevance of either the mathematics or the science they learn in school.

Mathematicians are already painfully aware that most school children find little connection between reality and classroom mathematics. Scientists see the same fate ahead for grade school science unless a determined effort is made to prevent it. I think that, in adopting the idea of an integrated curriculum, everyone of us at last summer's conference was primarily motivated by the desire to keep school work relevant.

New Thinking Required

We were, of course, aware that integration of science and mathematics could hardly be accomplished overnight. It will require a great deal of new thinking at the philosophical level and much experimentation. It will change the requirements for teacher training and alter the basic system of instruction. But, no matter how serious these difficulties may be, I am quite sure that we made a correct decision.

Even if integration cannot be accomplished within any reasonable time span, it provides a proper philosophical setting in which to plan curricular changes. When scientists and mathematicians are planning separate programs, efforts at coordination will always consist of one side asking and the other side changing —*maybe*. When an integrated curriculum is under consideration, a genuine spirit of cooperation is possible. At the conference I felt that the philosophical barriers which divide scientists from mathematicians began to melt away as soon as we had accepted the idea of integration.

We conceived of the integrated math-science curriculum as being organized along lines quite different from the present mathematics curriculum. Instead of a textbook which dominates the classroom and forces every child to go at the same pace, we thought in terms of a large number of small units, many more than any one child could handle. Children would study appropriate units either alone or in small groups. Teachers would help children over the rough spots and choose the units which seem best for each child. Teachers' manuals would be directed largely at helping the teacher select appropriate sequences for the pupils.

That kind of system offers a fair degree of individualization of instruction and restores to the teacher a large measure of responsibility for the choice of material covered. Such a system is now in day-to-day operation in many English schools.

The units themselves must encompass a broad spectrum of both content and outlook. They must range from purely mathematical questions handled from a strictly mathematical point of view to wholly nonquantitative science handled from a strictly scientific point of view, with all possible admixtures in between. Each unit must provide for the average child and yet be open-ended, to allow the interested student to go ahead.

In the middle of this spectrum will be the units which show mathematics and science standing side by side in the struggle to understand reality. These units will be the hardest to write and at the same time the most important since they will have the greatest relevance. It is at this point of confluence that the value of the math-science complementarity is most visible.

Classroom Applications

While such broad problems of pedagogical strategy generated some of the most spirited discussions at the conference, a great deal of energy was devoted to ideas more immediately related to the classroom. Because mathematics has long been a major topic in grade schools while science is just beginning to receive comparable attention, we had a much clearer idea of what can be accomplished with children in areas near to mathematics than in those close to science.

In its 1963 report, *Goals for School Mathematics,* the Cambridge Conference on School Mathematics proposed that "the objective for mathematics instruction in the elementary grades is familiarity with the real number system and the main ideas of geometry." If any such brief phrase could describe the objective envisaged by last summer's conference, it would be "familiarity with the quantitative view of the world." By the quantitative view of the world, I mean a natural tendency to seek out ways of measuring things relevant to any kind of intellectual problem and to inquire into the relationships which may exist between the results.

Mathematicians will see the latter as an enlargement of the 1963 objective in which the purely mathematical concept of the real number system is replaced by its application to the world around us. The scientists, I believe, favored this objective because of an overwhelming feeling that the lack of the quantitative outlook is the greatest barrier to science teaching today. Everyone had felt this lack, from those who had much experience with science teaching in elementary schools to those whose only experience was with college students.

The Crucial Question

Whether or not we can successfully teach the quantitative point of view is the crucial unknown in plans for an integrated math-science curriculum. I am optimistic on this point, but there are many who disagree. Indeed, it has been argued that quantitative science is impossible at the elementary school level

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and, therefore, there is no interface between mathematics and science instruction, at least until high school.

According to this view, only quantitative or descriptive science is appropriate in the elementary school. Most of the scientists at last summer's conference, however, felt that a totally descriptive approach misses the whole spirit of modern science. Perhaps the most important lesson to be learned from science as a whole is that our ability to understand any phenomenon is enormously increased if we can only find a way to measure it.

We are very much impressed by the work carried on in England by the Nuffield Foundation. They have started children off in the first grade collecting numbers and presenting them in various graphical forms, as seen in the Nuffield Project booklet, *Pictorial Representation*. Children apparently enjoy this activity and continue such quantitative work not only in science and mathematics classes but also in social studies, history, or wherever appropriate data may appear.

Although there has been a lot of experimental work done in this country which involves the use of graphs by young children, none of it seemed as broad in scope as that of the Nuffield Foundation. Their experience encouraged us to believe that the quantitative outlook can be taught. If this proves to be correct, it will have profound implications for both mathematics and science teaching. The suggestions that follow are both dependent on and part of the effort to instill an easy familiarity with matters quantitative.

Basics of Measurement

If we expect children to measure various continuous quantities and to handle the results, then we must make a much more conscientious effort to teach the basic ideas of measurement. We now focus almost all our effort in the first few grades on discrete interpretations of arithmetic; that is, on problems that are basically counting. Children learn that "$4 + 5 = 9$" is applicable in the form "4 apples and 5 more apples make 9 apples," and that it applies equally well to "4 elephants and 5 more" or to "4 astronauts and 5 more."

But how many second-grade children can answer the question, "If you glue a 4-inch stick and a 5-inch stick end to end, how long will the resulting stick be?" That addition has anything to do with this problem isn't obvious to a seven-year-old who has had experience only with discrete problems. Less than five percent of the pages of one popular workbook for first grade are devoted to continuous interpretations of numbers such as lengths or capacities. We must certainly plan to give these continuous applications something approximating equal time.

Children are taught to round off measurements from the beginning; it seems much better to teach them to report measurements in interval form. For example, they should say "This stick is between 4 and 5 inches long" instead of "This stick is 5 inches long to the nearest inch." The latter means, of course, this stick is between 4 1/2 and 5 1/2 inches, but the wording makes it very easy to lose track of the phrase "to the nearest inch."

When measurements are reported as intervals, the possibility of errors of measurement is always apparent, even though no further attention is given to them. While this approach seems to be most relevant to science, it has profound importance for pure mathematics as well. One of the hardest things to get over in a first-year calculus course is that numbers are often computed through a process of locating them within smaller and smaller intervals.

Statistics and Probability

A much larger change will be needed to introduce work in statistics. By statistics I do not mean a collection of numerical facts, but the science of organizing data and making decisions or predictions through the analysis of data. One might say that statistics is the quantitative science of learning by experience. The mathematics underlying this science is known as probability theory.

All of us make statistical decisions every day. Suppose we put the cat out before dinner because we are afraid he may jump on the table in the middle of the meal. Probably the cat hasn't always jumped on the table when he was allowed in at dinner, but it has happened often enough (especially when we are having fish) to convince us that we had better put him out. To put the cat out is a statistical decision based on our estimates of his probable behavior and the inconvenience it might cause, balanced against whatever inconvenience there may be in finding the cat and the pangs of conscience we may feel about banishing him to the cold.

In this situation the data were considered only informally because they were clear and (probably) overwhelmingly strong in support of our decision. Statistics has a serious role to play when the data are either weak or obscure. If we sample a dozen voters in a municipal election and find that seven prefer candidate A to candidate B, the data are clear enough but the small sample size makes them very weak. We can hardly be convinced by these data that A is a shoo-in. Statistics will help us decide what credence we should put in the results of the poll.

In a national election, however, if we have early returns from a few hundred voting precincts unevenly scattered about the eastern part of the country, we can with the aid of a computer make an excellent prediction of the final outcome. In this case the data are extremely strong but their message cannot be read with any certainty without a systematic analysis which takes into account the results in these precincts in prior elections.

Appropriate Objectives

Work in statistics for the average child should not aim at mastery of the most refined statistical techniques. More appropriate objectives are for each child to understand that (1) detailed analysis of data can in many cases lead to sounder decisions than casual inspection can lead to and that (2) in any such analysis one must constantly guard against unsuspected sources of bias.

The second of these objectives can only be attained through experience with actual problems. To achieve the first we should teach techniques sufficiently powerful to exhibit a clear advantage over naive analysis but based on an understandable mathematical theory. Analysis based on normal distributions, correlation coefficients, and a student's t-test require far too much probability theory. So-called nonparametric methods, based largely on the ordering of data, appear to be more promising.

A sketch was prepared at the conference for a program in probability and statistics which should serve at least as a start toward a viable school curriculum. It begins in the primary grades with finding the median (middle number when the data are arranged in order of size) of a data distribution and works up to comparing samples from different populations by examining their ordering (Wilcoxon test) in junior high school. Note that even a first-grade child can find the median height of the pupils in his classroom, whereas to find the average height would be several years beyond his power.

This leaves more traditional topics like normal distributions and the standard deviation to be taught in high school, presumably on an elective basis as they are now in some places. These courses could, of course, count on a much more sophisticated attitude toward statistics and therefore go farther than they do now.

Ratio and Proportion

There are significant areas of science in which the main mathematical tools are ratio and proportion. The director of a well-known science curriculum project stated that a major breakthrough in the teaching of science in high school would occur if only the students could master ratio and proportion. In the course of discussion of this point it became clear that proportion could be attacked by means of graphs very early in grade school.

As I have already remarked, there is ample evidence that children can use ordinary rectangular graph paper in grade one. If we graph the heights against the weights of the children in the class, the results will be fairly hodgepodge. But if we graph the lengths against the weights of small dowels cut from a single long dowel, the heights will be very nearly proportional to the weights so the graphed points will lie on a straight line. There are any number of similar experiments. Children will soon learn to look out for the possibility that their data points are on a straight line, and when they detect this phenomenon they can use it to make predictions.

In the height-weight experiment with dowels, after they find the line along which the graphed points lie, they can use it to predict the weight of a dowel of known length or the length of a dowel of known weight. Only the simplest ratios, like two to one, can be treated in the early grades as long as the method is one of pure calculation. But with graphs no ratio is more complicated than any other. Here again we see the possibility of doing significant work in the primary grades with a topic usually reserved for much older children.

It must be admitted that the graphical method does not lead to exact answers. If you compute by plotting points and drawing lines, you will probably conclude that 8:13 = 13:21 although this is not correct. I do not regard this as a disadvantage. One of the mistakes of classical mathematics instruction was its concentration on the unique "right" answer to such a degree that students never learned the value of an estimate. While I certainly don't derogate the ability to find the unique right answer, when one exists, I urge once again the importance of estimation, even in the course of finding an exact answer.

Moreover, it is essential to the quantitative point of view to recognize when an answer is necessarily approximate—as, for example, when the data entering the calculation are themselves approximate. The graphical method of calculating with proportions is not likely to give any child a wrong impression of what he is doing. On the contrary it ought to give him a much better feeling for the meaning of the exact calculations when he learns to make them.

Graphing Functions

Some experiments will lead to data points which seem to fall on a curve. These will lead naturally to the general concept of function, of which proportion is really a special use. While the idea of function has been included in recent mathematics curricula, it loses some of its vigor when approached as a purely mathematical matter using only examples which can be precisely treated at the grade-school level.

Even though there is necessarily some imprecision in empirical functions, the approach using graphs seems extremely valuable. It is not at all implausible that work on periodic functions should begin in the elementary school since there are many easily observable, approximately periodic functions in nature; for example, the lengths of shadows vary through the day but repeat the same variation rather closely from one day to the next.

If all of these ideas were introduced separately into the curriculum, they would certainly take too large a share of school time. But within an integrated math-science curriculum, all these topics will overlap with one another and with traditional arithmetic and with the topics now being introduced in science programs. As boys and girls learn to handle quantitative data from their own observations, it will be quite natural for them to analyze these data; the analysis will provide arithmetic drill while making both science and mathematics alive.

I wish I had space in which to describe some of the science experiments that were proposed at the Pine Manor Conference. They ranged from testing material strengths to testing the effectiveness of fertilizer; from growing yeasts to "growing" a small shell out of modelling clay. At one point we built a kaleidoscope large enough to get inside of. All the experiments had something to offer to both science and mathematics.

It will take a great deal of work to get these proposals and more like them tested in the classroom and ready for publication as reliable experiments to be done by the average child. It will take even more work to give them the coherence required for an integrated math-science program. Yet it seems to be a worthy effort and I feel sure that this effort will be made by the only group that can really be effective—the teachers.

Teachers will need support from many groups within our society—technical experts, publishers, politicians, parents, principals. As science and technology loom ever more important in our society we must make sure that this trend is properly reflected in our educational system. Education deserves our best efforts; it is our most direct way of influencing the future.

Many innovations in teaching methods have found application in mathematics classes. The authors of this article show what some of these innovations are and how they might be used.

Modern Teaching Methods for Modern Mathematics

WALLACE H. GEISZ
LEROY SACHS
ROBERT WENDT

SOONER or later every mathematics teacher approaches a topic in his course which he dreads, either because he doesn't understand it too well or because he has difficulty getting it across to his students. If several teachers are teaching the same course and there is an opportunity for all the students to meet at one time, then each teacher can present the topic in which he feels most qualified. These large-group presentations can then be followed by small-group problem-solving sessions.

Small groups should be limited to 25 students. The size of the large group would be determined by the facilities available. Auditoriums can be used, although these are not the best rooms for learning. Each student can be provided with a lapboard for taking notes. A theater-style room such as a band or chorus room which seats 100 to 150 students is ideal. Closed circuit TV could be used with several rooms.

At Clayton we are carrying on a team teaching program in elementary algebra (110 students, two teachers), and in advanced algebra and trigonometry (60 students, two teachers). Each stu-

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dent attends a large-group meeting once a week and a small-group meeting of 20 students three days a week. In addition, individual help is available in the mathematics resource center.

It takes a gifted teacher to lecture to an algebra class of 110 squirming ninth graders for 45 minutes. Their interest span, on the average, is less than five minutes. There is no chance for the usual give-and-take of classroom discussion. Standing at the overhead projector, a teacher often wonders whether he is getting an idea across. One way to find out is to ask a few simple multiple-choice questions, and then ask for a show of hands on the different answers.

For example:

> Two trains start toward each other from cities 300 miles apart, one going 20 miles per hour faster than the other. If they pass each other in three hours, how fast was each train going? Suppose we let r be the rate of the slower train. What can we let represent the rate of the faster train?
>
> How many of you say r?
> How many say 20?
> How many say $r - 20$?
> How many say $r + 20$?
> How many aren't sure?

In a similar way, the equation can be developed, and then the students can be asked to solve the equation. The teacher can give several carefully selected choices for solutions to the equation. One choice should always be: "How many didn't get any of these answers?" Sometimes it is a good idea to leave out the correct solution as a choice.

Grouping Arrangements

The other teachers should always be present at the large-group presentations in order to gauge student reaction and to make suggestions later for revising materials. Being present also gives them an opportunity to know exactly what material has been covered and how. They can then avoid unnecessary repetition in their own problem-solving sections. One of the difficult things about team teaching is accepting the fact that students do learn something in large groups. Problem-solving sessions need only be devoted to solving problems.

Every teacher should take part in planning the course. One advantage to team teaching is getting an agreement on what topics should be taught. Each student is then assured of studying the same basic materials no matter what section he is in. There is still room for individual differences among teacher and students. Testing, too, can be uniform. One test can be given to the whole group, with each teacher contributing his talents in constructing the test. Again, questions can be varied to provide for individual differences. Basic assignments can also be agreed upon.

If the students are grouped homogeneously in small sections, each teacher can add to the basic assignment those problems he thinks his class can handle. This means that the brighter student is challenged by covering topics in greater depth, rather than by acceleration. If acceleration is desired, then the honors students must be separated from the whole group.

In homogeneous groups it is important to remember that the less talented student needs to be in the smaller group; the brighter student can more easily survive in the larger group. Unfortunately, the reverse has too often been the case. The slower student needs individual help for both understanding and motivation. He hesitates to come to his teacher for help, so an opportunity must be provided for the teacher to work with him on a regular basis. His group also needs a different type of problem-solving program, and more attention needs to be paid to drill.

Resource Center

A mathematics resource center should be in operation in every school every hour of the day. This center should have the usual equipment, books, and audiovisual aids. It should be manned at all times by a mathematics teacher or a capable tutor or teacher aide. Students should be encouraged to go there to do their mathematics assignments, so that help is immediately available if they need it. If a teacher aide is in charge, he could proctor make-up tests, and supervise use of audiovisual equipment. If a video-tape machine is available, it can be used for reviewing large-group lectures. For the teacher, manning the resource center can be a rewarding experience. He gets to know what is going on in other classes and what problems other teachers have to face. He gets acquainted with all kinds of students. At Clayton, we find that we can keep the resource center in operation almost all the time by having each teacher spend about four hours a week in it.

Research on team teaching has not shown that it offers any great advantage to the student; also team teaching presents problems in scheduling. But it gives teachers a chance to work together and to draw on each other's experience. Needless to say, teachers working on a team should be compatible. In schools where the supply of good mathematics teachers is short, team teaching provides the student with an opportunity to study under the best teacher available.

Nongraded Instruction

If the suggestions in the report of the Cambridge Conference on School Mathematics are to be implemented, it will certainly be necessary for the secondary school program to drop its lockstep approach. Students who can master elementary algebra in 10 weeks should not be forced to sit through 40 weeks of it. On the other hand, there is no reason why a student should receive a failing mark for a whole year's work and have to repeat the entire course even though he understands the first 10 weeks' work. Some schools are dividing their courses into smaller units, and students can proceed at their own rate, repeating a unit when necessary.

Programed Instruction

At about the same time the revolution in mathematics began, a number of projects in programing mathematics learning were undertaken. A great deal of programed material has been published, some of it good and much of it bad. While programed instruction has not gained widespread acceptance as classroom procedure, it does have some uses. For example, programed instruction has a big advantage in providing immediate reinforcement—a student knows when he is doing well.

One big problem with it has been student motivation. Students initially like the novelty of working alone, but unless this work is coupled with frequent teacher conferences the novelty soon wears off. The less talented student who has trouble with mathematics most likely also has trouble with reading. Unless materials are written carefully in simple language with plenty of repetition, the slow learner will have difficulty with them and will soon give up. On the other hand, the brighter student can proceed at a much faster pace.

At Clayton we have used programed materials occasionally to allow transfer students and "late-bloomers" to complete requirements for advanced courses. We have also used them for brighter students who have been out of school for long periods because of illness. Such materials are available commercially for all of the usual secondary school courses.

Computer Mathematics

The computer now is common in large industries, businesses, and universities, and within the next decade or two it is likely to become an important tool in most public school districts of the United States. This growth in computer usage will be a consequence of the great variety of services which the computer can render, the great reduction in costs of these services, and the increased simplicity of programing.

Perhaps the most glamorous but presently least developed of computer applications is in computer-assisted instruction (CAI). Possessing many of the features of programed instruction, CAI could become an important method of individualizing instruction. Experimentation by Patrick Suppes at Stanford University has shown that pupils of all ages are highly motivated to pursue a program of studies through the operation of a terminal connected to a huge computer. The International Business Machines Corporation has begun an extensive research project in CAI, but much remains to be done.

The success of computer-assisted instruction will depend a great deal on the quality of the "software" (written programs) developed. It is difficult to write a program which takes into account all possible responses a student might give to a question. Missing, too, is the interplay of class discussion, although students like to "talk" to the computer. Again, the advantage of this type of instruction is that each student gets an immediate evaluation of his response; he doesn't have to wait until the teacher is available. If and when both the hardware and software are available at economically feasible prices, some combination of teacher-computer instruction will probably prove to be most effective.

Another application of computers in education is in the area of data processing and computing. Any district of any size has the task of storage and manipulation of a large amount of information about students, teachers, financial matters, inventories, etc. Information retrieval from a computer may become as common as information retrieval from the book-filled library of today. Computers can be used in many areas in which data now are just lost in the paper records because too much human effort is required to make the information useful.

School administrators should be very interested in the new developments in computer programing and related computer mathematics as areas of study for high school students. It is now known that many high school students can successfully learn to program a computer. The development of the time-sharing computer system also brings computer laboratory experience to a cost of between $300 and $400 a month, and the operation of computer terminals is simple enough to permit junior high school students to learn to use them. Whether or not the junior high program contains genuine problems for which computer solutions are desired is open to question. But students in high school biology, chemistry, physics, and mathematics can make good use of the computer. Students who have learned computer skills in high school often have an advantage when reaching college. For instance, some students who have studied computer mathematics at Clayton have been able to obtain part-time employment related to computers and others have made good use of their skills in college. There is a very good article entitled "Computers: Their Past, Present, and Future" by Donald D. Spencer in the January 1968 issue of *The Mathematics Teacher*.

Independent Study

Independent study is a phrase often heard in schools which have adopted or are thinking of adopting some sort of flexible scheduling. It is also being experimented with in schools which have traditional schedules but which feel the lock-step method of teaching has many drawbacks. In such schools, classes are usually organized so that each student pursues the subject matter at his own rate, taking tests on specified units whenever he is ready. The teacher is present to give individual help.

In some cases a student is not allowed to go into the next unit unless he has achieved a specified minimum grade on a test. The hope is that brighter students will not be held back, but may go faster and perhaps deeper into the subject matter. At the same time, the slower student can continue to study a given topic until he has achieved a degree of mastery in it, rather than being pulled along into the next unit even though he has little or no understanding of the unit just finished, a process which compounds the difficulties as the year progresses.

In practice these methods have not always been successful. Students soon get widely separated in levels of progress, and it is impossible for the teacher to hold a class discussion on a given topic. While there are many students who do not need class discussions to learn, there are many others who find it very difficult to learn mathematical ideas on their own by studying a textbook. There are many who may be able to learn basic techniques, but who will miss many of the implications of the principles involved. In other words, they may learn how but not necessarily why.

Teachers also find it personally not satisfying to spend their days as consultants. They miss the give-and-take of class discussion, and very likely there also is a loss to the students in this regard. Frequently the question asked by one student may open doors for others, and the teacher may also bring up questions that most students would not think of when left alone.

In schools with flexible scheduling, independent study has a different meaning. Students spend less time in the classroom and thus have more free time. Students may be expected to progress at the same rate from unit to unit, and there are class discussions and lectures. But the student is expected to take the responsibility of continuing investigations which are started in class, and of using materials such as textbooks, transparencies for overhead projectors, filmstrips, and whatever else may be available. These might be kept in a resource center, or if none is available, in the teacher's room.

With those methods a motivated student is able to achieve more in less time, and a slower student will be given the basic material in class and will have opportunities to get outside help during his free time when he needs it. Probably the biggest difficulty with this approach is in getting students to accept the responsibilities involved. For those who do, however, the rewards can be great.

Library assignments in mathematics are another means of getting students to study independently. If a school library has a good collection of mathematics reference books, specific short-answer questions can be given which require the use of such references. The purpose is not only to elicit specific information from students but also to get them to look into these books. They might find something interesting in books that too often gather dust on the library shelf. A good bibliography of mathematics reference books is available from the National Council of Teachers of Mathematics.

Use of Audiovisual Aids

One change that is evident in many mathematics classrooms is the use of the overhead projector. Many mathematics teachers used to maintain stoutly (and some still do) that the only equipment needed to teach mathematics was a piece of chalk and a chalkboard. But the overhead projector has offered opportunities not available with those traditional items. Some teachers use the projector instead of a chalkboard for everyday lectures and discussion. Such use has at least two advantages: (1) the teacher can face the class at all times, and (2) by using an elevated screen everyone can see clearly throughout the discussion, which is not always possible when using a chalkboard, particularly when lengthy problems are being worked.

Special advantages occur for specific instances. For example, a permanent grid can be prepared on a transparency for use with problems involving graphs. When a graph needs to be illustrated it is a matter of seconds to put this transparency in place and proceed with the grease pencil. The traditional method, of course, was to have a grid painted permanently on the chalkboard, which takes up valuable space, or to use a perforated graph stencil to put a grid on the board temporarily with chalk dust. This takes extra time, and it must be done each time a new graph is to be drawn.

But the overhead projector is valuable not only as a replacement for the chalkboard. Permanent color transparencies and overlays can be bought to illustrate many basic principles. For much less money, printed originals can be purchased from which transparencies can be made. These can be used in class discussions, and they can also be made available to students for independent study when time and facilities permit. For those schools whose facilities and scheduling practices allow students to use such materials, a variety of transparencies might be made available. For instance, geometric proofs might be prepared for quick checking of an assignment by the student himself.

The use of movies and filmstrips during class time seems to have limited advantages. Many of the films and filmstrips available cover very limited material and often in a manner not entirely compatible with the text being used, thus making such activity an uneconomical use of time. Again, however, if the facilities and time are available for out-of-class use, good films and filmstrips may be used by individual students or small groups on their own time to reinforce or supplement ideas discussed in class. This can become quite expensive, however, if these materials are bought, and very careful planning is necessary if they are to be rented.

Television

Closed circuit television can be used with team teaching or to bring special lectures to widely scattered groups. Some five educational TV programs in mathematics are available at local educational TV stations across the country. Each classroom teacher who uses such a program should have program notes ahead of time, and should be prepared to follow each program with a class discussion.

Use of Teacher Aides

The use of noncertificated teacher aides, or paraprofessionals, is being thought of more and more as class sizes grow and school budgets soar. This sort of help is more economical to provide than larger teacher staff, and in the right circumstances, it can enable more students to be served. A clear distinction should be drawn between an aide and a clerk or stenographer. In many instances an individual may serve in both capacities at once, but the duties and qualifications are different.

In the secondary field some subject areas lend themselves to the use of paraprofessionals much better than others. Science teachers, for example, can make very good use of personnel in setting up labs, taking care of equipment and supplies, preparing slides, and so on. In mathematics the possibilities are much more limited. This is probably especially true in schools which have traditional schedules, where most of the teaching is done in a traditionally organized classroom, and the student spends most of his day in class. In such schools the teacher aide would be restricted to such activities as helping with drill work (especially in classes for slow learners), handling programed materials, preparing transparencies for use on an overhead projector, and checking homework if the training of the aide is sufficient for this. In some cases the aide might even help in grading tests or preparing tests, again depending on his background.

There are clerical duties that could also be done by such a helper; for example, typing tests. While this kind of duty may not need any special mathematical background, a familiarity with such things as the spacing needed for various kinds of problems and the language used is a great help.

Flexible Scheduling

The term "flexible scheduling" attempts to name almost any scheduling which differs from the common six- or seven-period day which repeats five times a week every week of the school year. Modular scheduling usually means forming class groups of various sizes and combining blocks of 15 or 20 minutes into variable length periods. Class size and period length are chosen in order to make most efficient use of activities through which learning is to take place. Following is an example of a typical week's program of a Clayton mathematics teacher:

The numeral in each space represents the number of students.

MOD*					
1	Calculus 10		Resource Center	Calculus 10	Calculus 10
2					
3	Beg. Algebra—Section 1 18		Beg. Algebra—Section 1 18		Beg. Algebra—Section 1 18
4					
5	Beg. Algebra—Section 2 22	Beg. Algebra—Section 2 20		Beg. Algebra—Section 2 22	
6			Honors Trig. 11		Trig.—Large Group 60
7	Trig.—Section 2 21	Trig.—Section 1 20		Trig.—Section 1 20	
8					
9	Trig.—Section 2 21	Resource Center	Trig.—Section 2 21		Trig.—Section 2 21
10					
11					
12					
13					
14	Beg. Algebra—Large Group 117	Calculus 10	Honors Trig. 11	Honors Trig. 11	Honors Trig. 11
15					
16					
17	Resource Center	Resource Center			
18					
19			Resource Center	Resource Center	
20					

* Each module is 20 minutes.

Some new schedules maintain a basic six- or seven-period day, but the schedules vary from day to day, and groups of such daily schedules repeat on a three-, four-, or five-day cycle. One feature of many of the flexible schedule forms is a relatively large amount of unscheduled time for both students and teachers. This is intended to give students the opportunity to use libraries and resource centers, to meet with teachers for individual conferences, and generally to use such time in a flexible manner.

In practice, this unscheduled time may be wasted by a large number of students. In Clayton High School some of the poorly motivated students have spent too much time relaxing in the student lounge. This kind of student appears to need to spend most of his time in scheduled classes and scheduled study groups. The freedom of students at Clayton High School to visit their teachers for help with mathematics problems has been used very little by students. They seem to hesitate to approach a teacher for such assistance, perhaps because it is considered a favor by the teacher rather than a responsibility of the student.

In order to make mathematics teachers more accessible, the Clayton mathematics teachers have scheduled themselves into a study room for about two-thirds of the total school time. The teacher in charge may not be the student's regular teacher, but the number of students visiting this room has increased slightly.

The small groups (8 to 10 students) and short periods (40 minutes) have been very successful features of the schedule for the less able students in mathematics. These students can be kept effectively involved in their studies during such sessions.

Flexible scheduling by itself can surely not be expected to bring the solution to all problems related to learning by students, but the search for new ways to arrange students and time should continue. The basic task is to establish objectives, but the methods used to reach accepted objectives must be evaluated regularly. Perhaps our methods can be improved by flexible scheduling as much as or more than they have by things such as visual aids, modern textbooks, good libraries, and well-designed classrooms.

Staffing a contemporary mathematics program is not as easy as finding teachers to handle traditional fare. But after reading this discussion of the many channels available for effecting staff improvement, there is little excuse for any school to have an inferior mathematics faculty.

Staffing Modern Programs

ALBERT P. SHULTE
DAVID W. WELLS

THROUGH modern mathematics programs, many students have discovered that learning mathematics is exciting and interesting. Various math topics are being taught earlier in the curriculum than before, and colleges attest that students enter with a much more thorough understanding of mathematics than was the case in the recent past. However, with these benefits have come problems, often centered on the preparation of the staff to teach mathematics in the "modern" spirit.

A great many teachers who were effective in the traditional courses need more preparation in order to teach modern mathematics with the same effectiveness. Some school districts have adopted new materials without providing for teacher preparation to teach these materials effectively. Some colleges and universities have been slow to modify their teacher education programs to prepare teachers to teach modern courses.

This article focuses on successful practices in staffing and staff development. Particular emphasis is given to preparing staff members to teach modern mathematics curricula, with attention to both pre-service and in-service activities. Careful consideration

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is given to the important problem of inducting traditionally prepared teachers into a modern program. Some of the resources available to school systems to help with problems of staffing and staff development are described. Professional activities which can provide additional insights into mathematics and into the learning process are included. Ways of disseminating information to the teachers in a school system are presented.

A good deal of the discussion which follows has to do with increasing the mathematical or pedagogical knowledge of the mathematics staff. It is, of course, true that additional knowledge about mathematics does not automatically produce a better teacher, but any good teacher will be even more effective in direct proportion to how much he knows about the subject he teaches. Similarly, given two teachers with equal knowledge of subject matter, the more effective of them will be the one with more effective teaching techniques. A most important consideration in staffing is the provision of opportunities for upgrading staff members in knowledge of subject matter and in techniques of teaching.

Local Resources

The Mathematics Coordinator

An excellent way to deal with problems of staffing in a school district is to have a competent mathematics coordinator responsible for the entire mathematics curriculum, K-12. (In larger school districts, this coordinator may need to have assistants responsible to him for particular levels, such as primary, intermediate, junior high and senior high.) The coordinator works with teachers individually or in groups to improve the mathematics curriculum. If in-service sessions are held, the coordinator may teach the classes himself, may arrange for an outside consultant to teach the classes, or may cooperate with such a consultant in teaching the classes. The coordinator keeps teachers informed about professional conferences and meetings and about extension courses offered by universities. He may arrange for extension courses to be taught in the district or for a college to give credit for a series of in-service sessions.

In a district that is too small for a full-time mathematics coordinator, a part-time mathematics coordinator frequently teaches one or two classes in addition. Or a district may share a coordinator with a nearby district.

The Principal

The school principal plays a major role in facilitating the professional growth of his mathematics staff. He may set up a professional library, so that teachers in all subject areas can benefit from it. He arranges for the attendance of teachers at appropriate in-service activities. When new programs are adopted, he insures that the staff will be effective in teaching these programs. He also makes sure that teachers undertaking innovative practices receive recognition.

The principal should be familiar with the major premises of the modern mathematics movement. In addition, he should have some information about the particular aims, objectives, and materials of the major curriculum groups in this field, such as the School Mathematics Study Group (SMSG) and the University of Illinois Committee on School Mathematics (UICSM).

Demonstration Teaching

All teachers benefit from seeing master teachers in action. The opportunity should be provided by bringing a number of teachers together to observe demonstration classes or by enabling teachers to observe other teachers in action in the district, perhaps in their own school, or in nearby districts.

Demonstration teaching is an excellent way to illustrate good practices in mathematics teaching. Several demonstration sessions at different grade levels may be a regular part of the staff development activity each year. As an extension of this, video tape can be used to record a demonstration session or an ordinary classroom session and to play it back at various times to groups of teachers. Many school districts are acquiring video-tape equipment, are training students to operate the equipment, and are using it for such purposes.

A drawback to the use of demonstration classes is that many teachers feel threatened by such situations, because they feel unable to provide the same type of exciting and active experience in their own classroom every day. The demonstration class is primarily a showcase of excellent technique, not a sample of what takes place in a classroom day after day. Another danger in a demonstration situation is that although students may appear to be learning a great deal, since they are responding actively and answering questions, it may be found by investigating the next day that they really have not learned as much as it seemed.

IN-SERVICE PLANNING

Effective in-service sessions may be organized on a grade-level basis, particularly if a new textbook series is being adopted. Alternatively, a series of in-service sessions might involve teachers from several different grade levels or classes, to consider a topic important at all these grades, such as approximation and estimation, coordinate geometry, or the role of proof. An in-service program may provide a broad overview of the mathematics program or may deal specifically with the material to be taught in the next few weeks. In any case, the activities must be related closely to the text series being used. The presentations should not be too theoretical; they must present ideas that teachers can apply in the classroom in the immediate future.

It is important that the person conducting the in-service session not only be an expert on the mathematics being discussed but also an able teacher. He should give ideas on how topics can be developed in the classroom, as well as develop the mathematical content. Methodology is as important as content—in fact, in the "modern" approach to mathematics teaching, some of the changes in teaching techniques are more striking than the changes in content. The teacher trainer should also emphasize the way to evaluate the learning which is taking place in a new program and the level of expectation to set for the students, among other things.

If manipulative aids are discussed, teachers should have these aids in their hands and the session may be conducted much the same as if the teachers were the students. It may be wise to facilitate this practice in using aids by grouping teachers in twos or threes.

In-service activities are usually more effective in the evening or during released time provided for an afternoon meeting rather than at the end of the school day. Teachers are tired at the end of the school day and tend to let down at that time. Many schools are out at 3:30 p.m. or earlier; providing an hour's released time would mean that an in-service session could start at 2:30 p.m. and be out around 4 p.m. Some schools with special situations have conducted successful in-service sessions starting at 8 a.m. or 8:30 a.m. and continuing for an hour and a half. In such cases, students are asked to report to school late on mornings when in-service sessions are scheduled.

Compensation

Many teachers in the past have participated in in-service activities without compensation. However, it has long been recognized that such activities as coaching, directing dramatic or musical activities outside the school day, or even taking tickets at sporting events requires time and effort beyond that devoted to classroom teaching. Compensation for those activities has been firmly established, even though they do not contribute to increasing teacher competence. When a teacher spends time and effort to improve his professional competence, it is desirable that he be compensated for this in some manner. Further, in these days of teacher negotiations, it is increasingly common to find contracts calling for additional pay for additional time spent in job-related activities.

Replacing a Program

If a whole school or an entire grade is adopting a modern program in place of a traditional curriculum, the only feasible approach is an in-service program carried on throughout the year. Such an in-service program is probably desirable at any time the elementary school adopts a new program, whether the previous program has been modern or not. If the previous materials were also modern, the emphasis of the in-service sessions should be on the particular point of view of the new materials, their sequence, the time at which certain concepts should be "nailed down," and on what material is nonessential and can safely be omitted under pressure of time.

At the secondary school level, mathematics teachers are more often specialists in the subject, so the emphasis need not be so much on content of new materials as on techniques of teaching and on relating techniques to content. Without such an emphasis, secondary teachers can often manage to use modern materials in a traditional fashion.

The teacher preparing to use a new curriculum should know not only about his or her own grade level, but about the background of previous courses from which students will come and the later courses which they will pass on to in higher grades. Each teacher should be aware of the mathematical learnings which take place in the grade preceding and the grade following the one he teaches, and should have the overall goals and aims of the mathematics curriculum in mind.

PROVIDING A RESOURCE PERSON

Some school systems have provided a resource person in mathematics in each school, to provide classroom teachers with rapid and continuing assistance. The resource person is knowledgeable in both mathematical content and teaching techniques, and is available to answer questions and solve problems which arise in presenting new material. In the secondary school, the resource person is usually the mathematics department chairman, while at the elementary school level it would be the person on the staff best qualified in mathematics. The need for resource persons in mathematics at the elementary school level is recognized by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America, which recommends that 20 percent of the elementary school teachers be given training equivalent to that desired of teachers for junior high school mathematics.

Some schools have adopted a "buddy" system for new teachers, where an experienced teacher is assigned to a new teacher, to help him or her adjust to the school and to the teaching situation. This system can produce benefits for both teachers involved. The new teacher frequently is better prepared in mathematics and can answer questions asked by the more experienced teacher. On the other hand, the more experienced teacher frequently has a great fund of ideas for enriching the classroom experience for students with activities, games, manipulative materials, and the like.

Regional Resources

THE INTERMEDIATE SCHOOL DISTRICT

Certain states have intermediate school districts, which provide local school districts with services in any or all of the areas of administration, educational media, instruction, special education, or data processing. Staffs of these intermediate districts frequently include consultants in mathematics, whose services are available to local school districts on request, to conduct in-service activities or for curriculum planning.

REGIONAL MATHEMATICS CONFERENCES

A number of regional mathematics conferences are held throughout the year under the sponsorship of a college or university, an intermediate school district, or a regional or state education or mathematics organization. Such conferences give teachers the opportunity to hear experts in various areas of mathematics and mathematics teaching, to see good instructional techniques in action, to hear teachers tell about their experiences with new programs or approaches, and to discuss problems of mathematics instruction with others of like interests. There are frequently displays of new texts and supplementary materials and equipment at these conferences.

Colleges and Universities

Colleges and universities provide a variety of services which can contribute to the professional growth of teachers, including of course, summer sessions in which a teacher may enroll for additional training.

A number of fellowship programs are available to secondary teachers desiring further training in mathematics. The National Science Foundation funds a number of institutes at colleges and universities around the country. These institutes consist of three basic types: (1) academic-year institutes, where participants attend classes on a full-time basis, with free tuition, stipend, and dependent allowance provided; (2) summer institutes, where participants attend a session of six, eight, or 10 weeks, with tuition, stipend, and dependent allowance; (3) in-service institutes, where individuals attend evenings or weekends, receiving tuition and mileage. The institutes range from those designed for the teacher with minimal mathematics background to those designed for teachers doing post-master's-degree work, and from those with primary emphasis on pure mathematics to those most concerned with teaching methods. Information about programs and the applicants for whom they are designed can be obtained by writing the National Science Foundation, 1800 G Street, N.W., Washington, D.C., 20550.

In addition to National Science Foundation fellowships, certain industrial organizations provide fellowships for selected mathematics teachers. For example, the Shell Oil Company provides Shell Merit Fellowships for study during the summer at Stanford University and Cornell University. Teachers with five years of teaching experience are eligible to apply.

In addition to summer sessions and institute programs, many colleges and universities offer extension courses in mathematics or mathematics education. These courses usually meet in the evening and in many cases are located in public school buildings, making it convenient for teachers in the immediate vicinity to attend.

It is frequently possible to arrange with a college or a university that persons attending an in-service series may receive college credit. The college or university is often willing to enter a joint experimental venture with a public school and to provide credit for such a venture, which may take the form of an institute or a workshop.

College and university faculty members may serve as consultants to a school system, speaking at a curriculum day, helping a group to plan a new venture in teaching, or teaching demonstration lessons.

State and National Resources

Many states have state supervisors in mathematics, who are available on call to help a school district improve its mathematics program.

State and national mathematics organizations sponsor professional conferences, which provide the same sort of opportunities as do the regional conferences previously mentioned. However, these conferences are usually more ambitious, featuring more sessions and more exhibits, and often taking place over a longer span of time. The National Council of Teachers of Mathematics, for example, holds an annual meeting each year in a different part of the United States. In addition, the NCTM sponsors each year a number of regional meetings around the country, so located that at least one meeting each year is within a relatively short distance of any teacher in the United States and Canada. Shorter NCTM meetings are held in July in connection with the NEA Convention, and in December in connection with the annual meeting of the American Association for the Advancement of Science. Each NCTM meeting features excellent speakers on a variety of topics of interest to elementary, junior high, senior high, and junior college mathematics teachers.

The National Council of Teachers of Mathematics is primarily interested in improving mathematics instruction at the elementary and secondary levels. The Mathematical Association of America, although primarily composed of college and university mathematicians, recognizes that instruction in mathematics at the secondary level affects instruction at the collegiate level. Thus, many individual presentations at the summer meeting and the annual meeting of the MAA concern the teaching of mathematics at the secondary level. The Central Association of Science and Mathematics Teachers, which is interested in the teaching of both mathematics and science, holds its annual meeting in the midwest on Thanksgiving Weekend. Its membership is national in scope.

All three of those organizations sponsor publications of interest to teachers. The NCTM publishes three excellent journals: *The Mathematics Teacher,* for secondary teachers; *The Arithmetic Teacher,* for teachers of grades K-8; and *The Mathematics Student Journal,* intended primarily for secondary school students, but a valuable resource for teachers as well. In addition, the NCTM publishes a wide variety of supplementary materials and an excellent series of yearbooks devoted to specific problems in mathematics instruction.

The MAA publishes *Mathematics Magazine,* aimed at the level of the first two years of college, and the *American Mathematical Monthly,* primarily directed toward the last two years of undergraduate instruction and the first two years of graduate instruction. The *Monthly* contains a regular column, "Mathematical Education Notes," which frequently discusses programs at the elementary and secondary school levels.

School Science and Mathematics is the journal of the Central Association of Science and Mathematics Teachers; it includes a number of mathematical articles each year. Two special features are an annual list of science and mathematics books suitable for children and young people, and a list of research studies (usually doctoral dissertations) in mathematics education during the previous year. Both lists are briefly annotated.

Implementing a Modern Program

The Buena Valley School System decided to start using a modern mathematics text series at all grade levels. Most of the teachers in the district were experienced but had minimal formal training in mathematics. The program up to then had been traditional, emphasizing mechanical proficiency rather than mathematical understanding.

The district recognized the importance of beginning to prepare the staff for the new curriculum at the moment when the decision was made to move into a modern program. Some members of the mathematics staff participated in selecting the textbook series which the school decided to adopt. The study of various series served an in-service function for the staff members involved, and at several mathematics department meetings, they reported on progress being made. When the new series was decided on, the department devoted a meeting to discussing the reasons for selecting this particular series. After the materials arrived, several other department meetings were devoted to discussing the texts.

The mathematics coordinator made sure that the teacher's editions of the new books were available to staff members before school closed in the spring. In the late summer, prior to the opening of school, the mathematics department held an intensive one-week workshop. The teachers were reimbursed for coming to the workshop. At this time, resource persons were brought in to talk about some of the new content, but particularly about the new approaches to teaching mathematics. One speaker emphasized discovery procedures, another spoke on the use of a mathematics laboratory, and another spoke about the need to set realistic expectations for the students. The mathematics coordinator conducted discussions of the first few chapters of the texts, discussing both content and method. A variety of means were devised to measure pupil understanding.

Throughout the year, in-service sessions were held on a biweekly basis, to discuss the materials which the teachers would be teaching in the near future. Demonstration lessons taught by expert teachers were arranged periodically during the year. The coordinator and other mathematics resource persons were available for help at the request of a teacher, and the coordinator made periodic classroom visits to observe the new program in progress. Meetings were held periodically to evaluate progress toward the goals of the new program.

Fitting New Teachers into a Modern Program

The Springfield schools have been teaching modern mathematics for several years; the school system carried out extensive in-service activities when the new curriculum was adopted. However, in the past year the school district has had to hire five mathematics teachers who had no background or teaching experience in modern mathematics.

The Springfield schools regularly hold an orientation program for new teachers prior to the opening of school. The mathematics coordinator used two days of this orientation period to familiarize the new teachers with the goals and emphases of the new materials. He also discussed with them the need to set realistic expectations and suggested the use of the discovery method and other "modern" teaching techniques.

The Springfield school administration would have set up an in-service course for the five new mathematics teachers, if necessary. However, because several nearby districts were using the same program and also needed to train new teachers, they cooperated with the Springfield district to provide a 10-week in-service course in the content and methods of the mathematics program.

Keeping Teachers Informed

Mathematics teachers need to be kept up-to-date on new developments in instructional materials and teaching aids. Attendance at professional conferences can be helpful and teachers who attend conferences can later report to other teachers who were not able to attend.

The mathematics coordinator in each district should have funds to purchase new books (if he cannot get complementary copies), to rent or buy new films and filmstrips, and to buy other instructional aids.

As has already been mentioned, a professional library should be provided in each school district, and demonstration teaching has also been pointed to as an excellent way to disseminate good practices in mathematics teaching.

A resource person in mathematics appointed in each elementary and secondary school and provided with time in which to study new developments could disseminate new information. Also, regular staff meetings can occasionally be conducted as workshop sessions in which teachers share special techniques that they have found to be particularly effective.

Many districts have one or two curriculum days during the year, which are an excellent opportunity to inform mathematics teachers about important developments. A speaker may discuss current trends in mathematics education, a panel of teachers may describe a new project, or a workshop session may be set up. Displays of new materials obtained by the mathematics coordinator might be set up so that teachers may browse through the materials.

How Adequate Are Teacher Training Programs?

The revolution in school mathematics came so rapidly that many college faculties, as well as elementary and secondary school teachers, were left behind. Of course, a great deal of the push for change in the mathematics curriculum came from university mathematicians and mathematics educators, but many institutions were staffed with faculty members who found it every bit as difficult to adapt to newer ways of teaching mathematics as did their public school counterparts.

Now that the "new mathematics" movement in the schools is roughly ten years old, it is appropriate to reflect on the progress that teacher training institutions have made towards preparing their students to be effective teachers in modern mathematics programs. Have the colleges caught up, and are they fulfilling their responsibility?

The Committee on the Undergraduate Program of the Mathematical Association of America has set forth guidelines and recommendations for the amount of mathematical training desirable for teachers at various levels of the school program. These recommendations have been followed to varying degrees by colleges and universities around the United States, and have been endorsed by a number of national and state curriculum groups.

Prospective elementary teachers are receiving a much more thorough mathematics training in college than was the case 10 or 15 years ago. Most new elementary teachers have a good grasp of the principles of place value, the structure of the number system, and the whole-number algorithms. They have a better understanding of the rational numbers than was general in previous years, although the rational number system is not as well understood as the whole number system. Elementary teachers in general have a much better understanding of mathematics than formerly.

Many elementary teachers have a special need for increased competence in the area of geometry. Geometry has become increasingly important in the elementary school mathematics curriculum; it is now included at all grade levels in elementary texts but is the topic most often slighted by the teachers. Much more needs to be done in pre-service training to educate elementary teachers to the role of geometry in the curriculum and to give them a basic understanding of geometric principles and concepts.

Since secondary teachers have usually been specialists in mathematics, they have traditionally received a much better mathematics education than elementary teachers. This is still the case. Moreover, most of the institute programs designed to orient teachers to modern materials and methods have been designed for the secondary teacher, so they have benefited to a much greater degree from the money which has been poured into teacher re-education.

Deficiencies

In spite of the improvement in the preparation of secondary mathematics teachers, there are some striking deficiencies. Probably the most serious lack in the secondary teacher training program is that of preparation for teaching the "second-track" student—the slow learner, low achiever, noncollege-aspiring student, general mathematics student, and so on. Teacher training programs usually spend little time preparing future teachers to teach these students; yet most first-year teachers spend most of their time teaching such classes, which are second in number among mathematics classes only to first-year algebra.

Not only have the colleges neglected preparation for teaching the slow learners in mathematics, but the "modern mathematics" movement has until recently paid little attention to them. Moreover, most mathematics teachers see the teaching of such classes as a necessary but undesirable part of their induction to teaching; they look forward to gaining seniority so that they may move into teaching only college-preparatory courses. The universities and colleges must educate their prospective teachers to the point of view that slow learners and low achievers can learn mathematics if properly presented, and that effective teaching of these students is challenging and requires expertise.

The subjects of probability and statistics are beginning to play a much more important role in the mathematics curriculum, from upper elementary grades through secondary school. The notion of dealing with uncertainty by mathematical procedures is important in everyday life, and is a necessary part of every student's education. However, most mathematics teachers have insufficient training in these subjects, particularly in teaching probability and statistics in an exciting way, using laboratory and experimental methods.

Another area in which secondary teachers are frequently ill-prepared is in providing for individual differences among their students. Considerably more attention needs to be given to this facet of teacher training, and effective techniques of providing for individual differences need to be presented.

Many secondary teachers have received no formal instruction in how to evaluate the learning of their students. Attention also needs to be given to the associated problem of setting realistic and appropriate standards of expectation for students.

Aids to mathematics learning range from the esoteric to the "gut level," as this article points out.

Materials of Instruction

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THE materials available for instruction in school mathematics are growing in quantity, and the variance in quality is large. The observations in this paper are provided to help the school mathematics instructor discriminate among possible choices of materials for instruction. It is not possible to mention all possible sources, even peripherally, so the important areas we have chosen to consider here are digital computers, calculus, the last year, standardized measuring instruments, practical mathematics, multisensory aids, and teaching for discovery.

Digital Computers

Probably no phenomenon during the current decade has directly or indirectly affected Americans more widely than the advent of the electronic digital computer. At present, computers are utilized in maintaining records, in controlling air and auto traffic, in medical research, in simulation, and in countless other areas. Although the effect the computer has had upon us so far is certainly powerful, even the conservative will admit that its impact in the future will be Herculean.

The reasons for including a computer science course in the secondary school have been well established: to provide students with (1) an understanding of the capabilities and limitations

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of the hardware, (2) a familiarity with the common parts and their functions, and (3) a computer language which permits the student to use the machinery in solving mathematical exercises. In other words, the computer should be interpreted as a means, not an end.

It is well established that the cost of operating a computer can be quite large, if not astronomical, for a single school. Probably the most satisfactory solution to the problem is a remote control console. Such an installation is basically a teletype machine accompanied with a telephone and attached to an already existing computing unit. This has the advantages of minimizing operational costs and affording the student the opportunity of manipulating a "real" electronic digital computer. This arrangement has been used, for example, by certain schools in Gary, Indiana, in connection with the Illinois Institute of Technology.

Pragmatic and theoretical considerations lead to rejection of the "toy" type of computer, which frequently is anything but maintenance-free and which inadequately provides students with the fundamentals associated with today's computers. There are many sources of excellent instructional materials commercially available,[1]* most of them relatively inexpensive.

Our employment of the computer will be thought of as a humble beginning by future generations. The exponential increase in marketability of the computer and its use is phenomenal. Today's students must at least know the basic capabilities of the computer.

Calculus

A course in calculus is most frequently initiated in college at the freshman or sophomore level. Similar courses are increasingly offered in the last two years of high school, and this trend will continue to increase.

Currently many texts are designed for teaching calculus in high school. Groups concerned with the question "Where should the teaching of calculus begin?" support the trend.[2] However, that question and the one which asks "What should be included in a beginning calculus course?" must be answered by the individual school or college in light of certain evidence.

What kinds of evidence, and particularly curricular evidence, should the school evaluate? First, the school must ascertain that

* Numbers in brackets refer to sources listed at the end of this article.

an adequate number of its students can profitably study a minimum of one year of calculus; that is to say, these students have the mathematical competencies and mathematical maturity by the end of their junior year in high school which their predecessors of one decade ago had acquired by the time of graduation from high school. Second, the school mathematics department must have a qualified teacher for the course. Enthusiasm and dedication are necessary but by themselves insufficient requirements for such a teacher, who must be competent in mathematics.

If a school has suitable students and teacher for a course in calculus, if the school's enrollment is about 1,000, and if it offers a comprehensive instructional program, then it presumably can provide a quality course in calculus. Another rule of thumb is that from four to 10 percent of the seniors in an academic high school should probably be enrolled in a calculus course.

An additional consideration is whether the school should offer a two-semester course in calculus alone or in calculus and analytic geometry. There are arguments supporting each alternative.[3] Obviously the choice made must be compatible with the philosophy of the mathematics department and the nature of prerequisite courses.

Selecting a Text

Once this is decided, appropriate instructional materials must be selected. Of the many possible texts, 90 percent at least could be eliminated on the basis of their inadequate treatment of basic content. A text should be selected which provides mathematically accurate and concise definitions, a minimum of undefined words, "gut level" kinds of axioms, and mathematically appropriate theorems. Also, materials for instruction in calculus should provide the student the opportunity to utilize his intuition, but not to the degree that it would detract from the rigor of frequent proofs and exercises.

In addition to some excellent calculus texts, there are a few excellent films. [4,5] Some of these might be considered as single-concept films and are produced without sound.

A significant number of high schools currently offer a semester course entitled "calculus." Also, many schools are providing courses which teach a two-, four-, or eight-week unit about calculus. Programs of this and similar size are questionable; and they often utilize a text devoted to many topics and lacking depth. Another equally deplorable choice of instructional material for these short programs is the text which fosters predominantly intuitional and rigorless mathematical instruction.

If a bona fide two-semester course in calculus or in calculus and analytic geometry cannot be provided because of lack of time, another alternative would be a four- to eight-week unit on limits or on limits and continuity. Such a unit should not be described as calculus, but it should include the rigorous treatment of this content area, which is part of a carefully designed calculus course. There are good materials available which are specifically designed for this purpose.[6,7] These materials supplement the teacher's efforts to teach content as opposed to teaching about it. Many other content areas are frequently taught for a four- to eight-week period or longer, and many people consider them more desirable.

Programed materials can also be utilized in teaching calculus. These are good within the limitations inherent in programed materials, but their utilization should be limited to the very bright and highly motivated student who cannot enroll in a formal calculus class or in some other substantial mathematics course.

The Last Year

"What should be the nature of the content of the last year of secondary school mathematics?" This question remains unanswered, or partially answered at best, for many mathematics departments.

As employed here, the last mathematics course offered in the secondary school presupposes detailed study of high school algebra, trigonometry, and geometry. With this mathematics background established, what are some plausible answers to the question and, in particular, what implications do these choices have with respect to materials for instruction?

The most important consideration in selecting mathematics content for the last year of school mathematics is that it provides for and encourages study of a topic in depth. In other words, the student becomes increasingly concerned with axioms, undefined terms, definitions, and theorems, and with the interdependence of structural elements as a manmade check-and-balance system. In brief, the student is "doing mathematics."

The content selected for this course should not, in general, utilize a college text as the main source of reference because of the repetition which would be encountered at a later date.

The previous discussion of the teaching of calculus in the secondary school hopefully has provided a guideline for selecting materials if calculus is chosen as the topic of study for the last course in school mathematics. Similarly, previous remarks relative to initiating a course in digital computers will be of assistance in selecting materials for instruction if that choice is deemed desirable.

If, on the other hand, a different topic is selected, several excellent pamphlets and booklets are excellent choices for the "main text."[8] There are also some fine textbooks, but a text which includes many topics and consequently does not provide adequate depth in any topic is undesirable. In summary, it is unquestionably better to use materials providing depth than those which "cover the waterfront" and consequently provide the student with a meager or superficial reference.

Specific content areas which might be considered for the last year of school mathematics as alternatives to calculus or the study of digital computers are many. Topics which lend themselves to study in depth and are compatible with the current philosophy of school mathematics include probability and/or statistics, algebra structures (groups, rings, integral domains and fields), linear algebra, and geometry. Of these, geometry is the only one infrequently selected. This is a rich area and should include those materials which provide the student with substantial structural geometry—in-depth study of Euclidean geometry or, probably more desirable, an axiomatic appraisal of several kinds of geometry.[9,10]

Although the foregoing topics are desirable alternatives for the last course in mathematics, this list definitely is not exhaustive. Depending upon such variables as the characteristics of the student body, individual teacher competencies, and the philosophy of the mathematics department, alternate areas of content will be selected.

A final suggestion for the terminating course in high school mathematics is a mathematics seminar. In such a course, highly motivated students study mathematics topics of their selection on an individual basis. An ideal class size would probably be somewhere between 10 to 20 students. Such a seminar would require a wealth of quality mathematics materials, including not only texts but additional materials about mathematics.[11]

The seminar develops a theme from simple to complex such as complex numbers to quaternions, one dimensional to n-dimensional geometry, plane to complex topological surfaces, etc. A word of caution is in order at this point. Such a seminar demands not only an extremely competent teacher with considerable knowledge of mathematics but, in addition, one who can continually motivate the students to effective work.

Standardized Measuring Instruments

Probably no aspect of schooling is more neglected than evaluation. Although mathematics is no exception, mathematics teachers do at least a satisfactory job of testing. To improve the use of standardized tests, a few observations seem appropriate.

Commercial tests which were produced before 1960 should be carefully scrutinized and evaluated before use. Testing instruments of this vintage and earlier are frequently intended to measure objectives which differ from and in some cases are dramatically opposed to contemporary objectives. This does not imply that all such instruments are outdated and inappropriate, but rather that critical appraisal of the objectives which the instrument is measuring is of vital importance.

Many mathematics teachers have found that students' scores on certain testing instruments are less than their predecessors if there had been a shift from a classic traditional program to one reflecting a more contemporary spirit. Results of this type have supported the inadequacy of many older commercial tests and simultaneously have motivated the publishing of testing instruments which measure objectives currently deemed desirable.

Probably the easiest and least valid means of identifying a contemporary test instrument is a cursory examination of the mathematical vocabulary employed. Those tests attempting to measure present-day objectives will appropriately employ such words as ray, identity element, set, and binary operation. Clearly, the selection of an instrument on the basis of the vocabulary used in it would be insufficient evidence, but in all probability this would be a necessary criterion.

Currently there are several publishers producing contemporary test instruments, such as the Stanford Achievement Tests.[12,13]

Samples of many instruments should be evaluated before one or two are selected. It is most important to determine the degree to which the objectives being measured by the test are the course objectives aimed at by the mathematics teacher. The data from the test must be in a form compatible for evaluation; for example, if the student marks the statement, "The real numbers are closed with respect to addition," little if any evaluation can be made of his response because the measure resulting did not identify what the student experienced in providing the measure.

In including commercial test instruments in the secondary mathematics program, such testing should probably occur only once or twice per year. One of the main reasons for using the commercial t ' instrument is to provide the student with a score which indicates his relative rank on a national basis. This information is desirable and necessary for use by several faculty members as well as by the student and possibly by his parents. However, the use of such scores to determine a significant portion of a student's six-week grade or course grade is questionable.

Practical Mathematics

It is frequently echoed that materials for mathematics instruction should be designed so that their content is practical. What does it mean for mathematics to be practical? The most frequent response to this question is that the student utilizes his knowledge of mathematics to solve a problem which occurs in his daily living.

Problem-solving situations are frequently manufactured superficially by the teacher, and they may misrepresent the mathematical problems encountered in the student's real world. Therefore, materials should be provided which honestly require mathematical skills which the students can presently employ in problem-solving situations. Also, materials should provide the student with those fundamental competencies which he most likely will utilize during his life. Materials for instruction should realistically identify the mathematics a student will probably apply now and that which he probably will apply in future. Practical mathematics of this nature should be considered as general education and should be included in mathematics courses required of all high school graduates.

What are some of the more obvious "gut level" topics which should be included in practical mathematics materials? The answers to this question depend upon such variables as density of population and predominant forms of employment. However, practical topics include rates of interest, dividends, insurance, stocks and bonds, measurement as commonly applicable, interpretation of probability and statistical data, and an appreciation of geometry. Most of these topics are adequately treated in most texts with the possible exception of probability and statistics, and many currently published texts are eliminating that deficiency.

The foregoing brief description of practical mathematics is acceptable for most people. On the other hand, to the mathematics teacher such a description is inadequate, not because it is inaccurate but because it is incomplete. It is incomplete in the sense that any mathematics content can be practical when it motivates the student to become mathematically hungry. Practical mathematical materials of this nature are vitally important in all secondary school mathematics in general, and in elective courses specifically. Such materials are plentiful and most frequently are in the form of a booklet or book different from a text.[14] These materials may be readily available, but knowledge of where to get them may not be readily available. Therefore, choice instructional materials should be retained for continual reference.

Multisensory Aids

Multisensory aids have been utilized for mathematics instruction for several years. Of the many excellent aids available, the overhead projector has probably made the greatest contemporary impact. This is supplemented by large numbers of packets of materials commercially designed for school mathematics instruction.[15] In fact, some packets are designed as supplementary to a particular text or series of texts. Many of these commercial overlays are quite good. However, the teacher-made transparencies are frequently very effective. If the school audiovisual department minimizes red tape, the teacher-made visuals are frequently superior to many of the commercial efforts. The coordinated use of dittoed student handouts and overlays of thermo-hectographs is most advantageous.

Most teachers of mathematics minimize the use of motion picture films in their teaching. This medium should be utilized more since the quality of current films has increased appreciably. Also, these films have been professionally evaluated, which has enhanced the probability that their mathematical and pedagogical quality is high. The mathematics teacher can further refine his classroom use of films by showing only a portion of a film on certain occasions, as evidenced by the surging popularity of the three-minute, single-concept film loop. Film loops specifically designed for school mathematics instruction are increasingly available and will undoubtedly be utilized by teachers.

Another aid particularly important in the teaching of mathematics is semipermanent chalk, which is available either white or colored. It remains on the chalkboard even when dusted off with an eraser which removes normal chalk dust, but is easily removed by washing with water. Note, however, that symbols recorded with semipermanent chalk become permanent on certain chalkboards. Therefore, test this magic chalk in a small area at the corner of your chalkboard before initially experimenting with an elaborately colored visual which could haunt you.

Like other materials for instruction, there is no unique procedure to be employed with respect to a specific material. However, the semipermanent chalk is effective for recording those symbols which are frequently utilized all day. Some examples are graphs (coordinate or polar), general equation of a family of equations, those steps which are repeated in an algorismic procedure, certain formulas, and geometric figures.

An instructional medium frequently receiving minimal attention in the mathematics classroom is the bulletin board, which can become worth its weight in gold. Today's schools are frequently adding an audiovisual specialist to the faculty, whose suggestions and services may become indispensable. Additional ideas for designing bulletin board displays in the mathematics classroom are available in pamphlets specifically designed for this purpose.[16,17] The desirability of student participation in designing and assembling a mathematics bulletin board can hardly be overemphasized.

Student building of physical models designed to clarify the explanation of a particular mathematical principle or application is valuable. The most significant value received by the student in projects of this nature is the quantity of mathematics which he must employ. Frequently much learning of mathematics is necessitated which he had not anticipated. Clearly, such building projects graphically point out to the student how fundamental principles can be used in the physical world. The teacher should be aware that it is the means to the completed project which provides the student the valuable mathematical experiences, not the end product. Therefore, it is desirable for students to build their own projects each year and to not retain models from previous classes.

Of many additional multisensory aids, two should be emphasized. The first is the plane table.[18] This extremely simple piece of surveying equipment can be inexpensively purchased or made by students. It is most durable and can be effectively used for instruction at all grade levels in the secondary school; basically it has all the instructional values of the expensive transit and accompanying equipment.

The second multisensory aid which more and more teachers are finding to be instructionally sound is paper folding.[19] Paper folding utilizes waxed paper and the student's ability to apply principles of geometric construction. This very inexpensive medium provides the geometry student an excellent means of increasing his mathematical intuition and an alternate vehicle to study constructions.

Multisensory aids should not necessarily be utilized just for the sake of using an aid, but only when they lend themselves to clarifying concepts. For example, it is difficult to conceive of a mathematics instructor teaching about inequalities in a beginning algebra course without employing an overhead projector. How better can a teacher discuss half planes with respect to equations and inequalities than by using transparencies and colored acetate?

Teaching for Discovery

Of the several hallmarks associated with the teaching of contemporary mathematics, no characteristic is more frequently referred to than teaching for discovery. Although current research supports the teaching for discovery, the evidence has not been overwhelming. Therefore, it would appear that materials and teaching techniques should provide opportunities for and encourage mathematical discoveries on the part of the student.

Since the basic ingredient in discovering generalizations is the recognition of patterns, material for instruction should foster the discovery of patterns.[20] At the same time, the material should not demand generalization by the student for which he has not yet acquired appropriate language skills. Both the frequency and magnitude of mathematical discoveries should increase from year to year as the student completes secondary school. Materials should reflect these kinds of opportunities for discovery. Instructional materials for the very capable students in particular should be written so as to necessitate a significant amount of discovery.[21] The most important consideration with respect to materials designed to promote mathematical discovery are the exercises in the basic text. The variation between the quality and quantity of exercises from text to text is large. Therefore, the selection of a text should reflect this important consideration.

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